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MARKOV MAINTENANCE MODELS WITH REPAIR

by

YUKIO HATOYAMA

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CHAPTER 1

INTRODUCTION

1.1. Statement of Problem

This paper considers the problem of finding optimal policies for several deteriorating systems. The systems discussed here include deteriorating machinery whose deterioration can be observed by a decision maker or any system that involves periodic repair. Because of the applicability of these systems, a number of authors have contributed to the development of the optimization theory for the discrete time maintenance models. Almost all of them are replacement models rather than repair models in the sense that an unlimited supply of new spares is available and the amount of time needed for the replacement or repair is negligibly small. In practice we often encounter the cases where there are a limited number of spare units and where the repair time of a machine is no longer negligible as compared with the length of its operating time. Repair time of a machine may even depend on its degree of deterioration. Several examples of such processes can be easily introduced. Aircraft repair problem of an airline company, automobile repair problem of a taxicab company and factory machine repair problem are some of them. In particular, the problem of determining the date and quantity of suits one should bring to a laundry for dry cleaning can be applicable if cleanliness of a suit can be somehow measured.

Since repair rather than replacement is often applicable, we have developed some mathematical models which incorporate maintenance with repair.

The discrete time Markov maintenance problem with repair can be stated in the following way. A machine is assumed to be operating over time with its condition deteriorating as time goes on. It is observed at discrete time intervals and is classified as being in one of $I+1$ states $\{0,1,2,\ldots,I\}$. State 0 represents a machine in perfect working order, and $\{1,2,\ldots,I-1\}$ represents intermediate states of deterioration leading to failure at state I. An operating machine can be sent to a repair shop at the beginning of each period just after the observation of its state, whereas a failed machine must be repaired. When a machine is being repaired, the number of time periods that it is unavailable is usually assumed to have a geometric distribution. A repaired machine becomes available in its best condition. If the repair decision is not chosen at the beginning of a period, the machine in the i-th operating condition keeps operating and at the beginning of the next period it moves to the i-th operating condition with stationary transition probability p_{ij}. An operating cost is charged while a machine is operating, and material and labor costs are charged when it is being repaired. These costs usually depend on the state of the machine. The objective is to find a repair policy which minimizes the total expected α-discounted cost, or the long-run expected average cost per unit period.

There is a simple repair rule called a control limit policy: Repair the machine if and only if its observed state is greater than or equal to some control limit i^*. Control limit rules are intuitively appealing, but are not optimal in all cases. Certain restrictions must be placed on both the cost structure and the transition probabilities governing deterioration. Chapter 2 investigates sufficient conditions under which an optimal policy is assured to be of a control limit form.

The aforementioned model had only one machine in the system. In Chapter 3 models with spare machines in the system are studied. In these models when a decision maker decides to repair an operating machine, it is sent to a repair shop immediately and replaced by a spare machine, if any is available. Replacement is assumed to take one period of time just for mathematical simplicity. Machines in the repair shop are independently repaired with geometric repair time distributions. The system fails when all the machines are in the repair shop and inoperative. A penalty cost is then assessed. Special emphasis is being placed on finding the sufficient conditions which result in the optimality of control limit policies of some kind.

Chapter 4 presents some recursive algorithms for calculating the optimal cost and its corresponding policy. Lastly, an explicit calculation of an optimal control limit policy is obtained for systems with few machines and with simple cost structures and transition probabilities.

1.2. Brief Survey of Discrete Time Maintenance Models

In this section we give a brief review of previous studies that relate to our work. For almost two decades there has been a great deal of interest in the study of maintenance problems because of their applicability in the practical world. Maintenance models are usually divided into discrete time models and continuous time models. As our study is in the category of discrete time models, continuous time maintenance models will not be taken into consideration here.

Discrete time maintenance models are the models which use the information regarding the state of the system such as the degree of deterioration of the unit in order to choose the best maintenance action at discrete time points. In 1963 Derman [7] introduced the basic model of this type. In his model, there is a unit in a system which is inspected periodically. After each inspection it is classified in one of $I+1$ states, $0,1,\ldots,I$, showing the degree of deterioration. There are two actions available at each period. If replacement action is taken, the unit is replaced by a new unit and the new unit begins to operate in its best condition at the beginning of the next period. If the action, not to replace, is chosen, the unit keeps operating. Then P_{ij} is the probability that the state of the unit moves from i to j in one unit of time. The cost to replace an operative unit and the cost of replacing an inoperative unit, i.e., a unit in state I, are introduced. The latter is assumed to be greater than the former. Derman showed the existence of a control limit policy optimizing the

total α-discounted cost and/or the long-run average cost under reasonable conditions on the transition probabilities.

There are several directions for extending Derman's basic model. Kolesar [14] extended the basic model by introducing state dependent operating cost without changing the basic conclusion of the model. Kalymon [11] further generalized the cost structure by allowing replacement costs to be stochastic, retaining the optimality of the control limit policies.

Another expansion of Derman's model with regard to the cost structure is the introduction of an inspection cost by Klein [13]. He assumed that the condition of the system is known only when it is inspected, which is costly. At each period the decision maker has two basic alternatives: to replace the system or to keep it. When the latter is selected, he must further decide the extent of repairs to be made and when to make the next inspection. He formulated the model as a LP problem. Derman [8] later showed that the optimal solution is a stationary policy. Eppen [10], Taylor [20] and Rosenfield [17] developed their models along this line with emphasis on trying to find simple types of policies which optimize their models.

Derman and Lieberman [9] considered a joint replacement and stocking problem, which was generalized by Ross [18] to allow for a deterioration of a unit from time to time. Ross also extended the state space to be continuous or denumerable. Another expansion of the basic model with respect to the state space was made by Wagner [22], who considered two modes of failure.

As was mentioned earlier, the distinction between repair and replacement was first considered by Klein [13]. In his model an immediate transition to the best state takes place when a replacement is made, while an immediate transition to one of the better states takes place (depending on the extent of the repair) when a repair is made. In 1973 Kao [12] introduced a semi-Markovian approach to the basic model. He allowed the time spent in each state before a transition takes place to be a random variable depending on the transition. Therefore in his model repair is no longer instantaneous but takes some random time. Kao found sufficient conditions for the optimality of control limit policies over the class of stationary non-randomized policies. Among them are state independent expected repair time and repair costs, which could be considered as restrictive conditions.

Derman [6] himself considered his basic model using a different objective function. He maximized the expected length of time between replacements subject to the conditions that the probabilities of replacement through certain undesirable states are bounded. Kolesar [15] showed the control limit optimality for this case.

1.3. Mathematical Background

Before proceeding further it is well to describe some basic terms and fundamental theorems frequently used in our report. A Markov decision process is discussed first since this theory is the primary approach used in each model formulation.

Suppose a system is observed at points of time $t = 0,1,\ldots$ and classified into one of a number of possible states labeled $0,1,\ldots,I$. After each observation of the state of the system, an action must be selected. The set of all possible actions is denoted by A, assumed finite. If action $a \in A$ is chosen when the state of the system is i at time t, a cost $C(i,a)$ is incurred, and the next state is determined according to

$$p\{X_{t+1} = j \mid X_0, a_0, X_1, a_1, \ldots, X_t = i,\ a_t = a\} = p_{ij}(a) ,$$

where X_t is the state of the system at time t, and a_t is the action chosen at t. A rule or policy is a prescription for taking actions at each time. It may be randomized or it may depend on the past history of the process. A policy is called stationary if it is nonrandomized, and the action it chooses at t depends only on the state of the system at t. Let $R = (f,f,\ldots)$ be a stationary policy whose action is $f(i)$ when the state of the system is i. Then it is known that a process modified by f is a Markov chain with the transition probability matix $\{p_{ij}(f(i))\}$.

Two criteria are of interest as a suitable cost to be minimized. The first criterion is the total expected α-discounted cost. Let $\alpha \in (0,1)$. The total α-discounted cost when a policy R is employed is represented by

$$V_{R,\alpha}(i) = E_R\left[\sum_{t=0}^{\infty} \alpha^t C(X_t, a_t) \mid X_0 = i\right] , \qquad 0 \le i \le I,$$

where E_R means the conditional expectation given that the policy R is employed. Let

$$V_\alpha(i) = \inf_R V_{R,\alpha}(i) , \qquad i = 0,1,\ldots,I.$$

The following theorems hold.

<u>Theorem 1.1</u> (see Ross [19]). V_α is the unique solution to

$$V_\alpha(i) = \min_a \{C(i,a) + \alpha \sum_{j=0}^{I} p_{ij}(a) V_\alpha(j)\}, \qquad i = 0,1,\ldots,I.$$

Furthermore, if $R_* = (f_\alpha, f_\alpha, \ldots)$ is the stationary policy satisfying

$$C(i,f_\alpha(i)) + \alpha \sum_{j=0}^{I} p_{ij}(f_\alpha(i)) V_\alpha(j)$$

$$= \min_a \{C(i,a) + \alpha \sum_{j=0}^{I} p_{ij}(a) V_\alpha(j)\}$$

for each i then

$$V_{R_*,\alpha}(i) = V_\alpha(i) , \qquad i = 0,1,\ldots,I.$$

The first part of the above theorem gives a functional equation satisfied by the optimal cost function V_α, while the last part guarantees the existence of an optimal α-discounted

policy. Furthermore it may be taken to be stationary as has been shown in Derman [5], Wagner [21] and Blackwell [3].

Theorem 1.2 (Bellman [2]). Let $V_\alpha(i,n)$ be the n period optimal expected α-discounted cost starting from state i. Then

$$V_\alpha(i) = \lim_{n\to\infty} V_\alpha(i,n) , \qquad i = 0,1,\ldots,I.$$

Theorem 1.2 shows the validity of formulating the infinite horizon problem as the n period problem, and then letting $n \to \infty$.

The long-run expected average cost when a policy R is employed is expressed as

$$V_R(i) = \lim_{T\to\infty} E_R[\frac{1}{T}\sum_{t=0}^{T-1} C(X_t,a_t)|X_0 = i], \qquad i = 0,1,\ldots,I.$$

The policy R^* is said to be average cost optimal if

$$V_{R^*}(i) = \min_R V_R(i), \qquad i = 0,1,\ldots,I .$$

The existence of such a policy is not guaranteed in general, but the following theorem gives a sufficient condition for the optimality of a stationary policy as well as for the existence of an optimal policy.

Let π be the class of all policies, and let π' be the class of all stationary policies.

Assumption 1. For all $R \in \pi'$, the modified Markov chain by R is irreducible.

Theorem 1.3 (see Ross [19]). If Assumption 1 holds, then there exists $R^* \in \pi'$ such that

$$V_{R^*}(i) = \min_{R \in \pi} V_R(i) = \min_{R \in \pi'} V_R(i) = \lim_{\alpha \to 1} (1-\alpha) V_\alpha(0).$$

Theorem 1.3 also shows that the optimal average cost is independent of the starting state of the process and is related to the total α-discounted cost. Therefore the results obtained for the total α-discounted cost can be often applied to the expected average cost case through this theorem. The irreducibility assumption, however, seems too strong for our models. Fortunately the next assumption suffices.

Assumption 2. For all $R \in \pi'$, there exists a special state, say i_s, which is accessible from every other state.

Theorem 1.4 (Ross [19]). Theorem 1.3 still holds after replacing Assumption 1 with Assumption 2.

In reliability theory where the state of a system is represented by being good or bad, increasing failure rate (IFR)

distributions play an important role (see Barlow [1]). An item with an IFR failure distribution has the property of aging and is more likely to fail the older it gets. Derman [7] developed the notion of an IFR distribution for a Markov chain with a finite number of states. Suppose there are I+1 states indexed to show the degree of deterioration. Then a Markov chain is said to be IFR if the higher the state the greater the chance of further deterioration. More precisely, a Markov chain with the transition matrix $\{p_{ij}\}$ is said to be IFR if $P_i(\cdot)$ is stochastically smaller than $P_{i+1}(\cdot)$ for $i = 0,1,\ldots,I-1$, and can be written as

$$P_i(\cdot) \subset P_{i+1}(\cdot), \qquad i = 0,1,\ldots,I-1,$$

where

$$P_i(k) = \sum_{j \leq k} p_{ij}, \qquad i = 0,1,\ldots,I.$$

Note that suppose $F(t)$ and $G(t)$ are distribution functions, $F(\cdot) \subset G(\cdot)$ if and only if $F(t) \geq G(t)$ for any t. Since we are concerned with maintenance problems, aging of a machine is a reasonable assumption, and hence, IFR distributions will be assumed in each of our models. Lastly a useful lemma is stated.

Lemma 1.5 (Derman [7]).

$$P_i(\cdot) \subset P_{i+1}(\cdot), \qquad i = 0,1,\ldots,I-1,$$

if and only if for every nondecreasing function $h(j)$, $j = 0,\ldots,I$, $H(i) = \sum_{j=0}^{I} p_{ij}\, h(j)$ is nondecreasing in i for $0 \leq i \leq I$.

CHAPTER 2

MAINTENANCE MODELS WITHOUT SPARE UNITS

In this chapter we present several machine repair models where there is only one machine in the system. The first section treats the special case where both the repair time and labor costs are independent of the state of the machine, while the last section treats the general case where they depend on the state of the machine. In each section obtaining sufficient conditions for the optimality of a control limit policy is of special interest. In the second section the results in the first section are shown to be applicable to the case where the repair time distribution is no longer a geometric distribution.

2.1. Special Case

Consider a discrete time machine repair problem which is schematically shown in Fig. 2.1. There is a machine in the system with its condition deteriorating as time goes on. It is observed at the beginning of each period, and is classified as being in one of I+1 ($I \geq 1$) states. State 0 represents

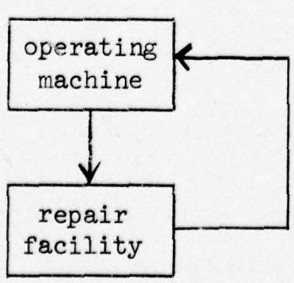

Figure 2.1: A Machine Repair System Without Spare Units.

a machine in its best condition, while $\{1,2,\ldots,I-1\}$ denote intermediate states of deterioration leading to failure at state I. Also a state I+1 is introduced, denoting that a machine is under repair. When a machine is operating, two choices are available just after each observation of its state by a decision maker: to let it keep operating, or to send it to the repair shop. If the former decision is selected, the state of the machine evolves from i to j in one unit of time according to the transition probability $p_{ij} \geq 0$. We assume $\sum_{k \leq I} p_{ik} = 1$ for any i throughout this paper. If the latter decision is chosen, the machine is immediately sent to the repair shop, and stays there until it is completely repaired. A failed machine must be repaired. The number of time periods taken to complete the repair work is assumed to have a geometric distribution independent of the state of the machine to be repaired. More precisely, let T_i be a repair time of a machine in state i. Then

$$P\{T_i = j\} = q(1-q)^{j-1}, \qquad j = 1,2, \ldots$$

Thus, a machine in the repair shop is completely repaired by the end of that period with probability q. Assume $0 < q \leq 1$. A repaired machine becomes available in its best condition.

The costs involved in this system are:

A(i): operating cost for a machine in state i $(0 \leq i \leq I)$.

C(i): material cost for repairing a machine in state i $(0 \leq i \leq I)$.

B: labor cost for repairing a machine per unit period.

Note that the statement that a failed machine must be repaired is satisfied if we set $A(I)$ to be infinity or a very large number. Material cost is assumed to be charged at the start of the repair work. The objective is to find a repair policy which minimizes the total α-discounted cost.

This system is a Markov decision process and hence, can be formulated as a dynamic programming problem. Let $V_\alpha(i;n)$ be the expected n period α-discounted minimum cost starting from state i at the beginning. Then $V_\alpha(i;n)$ satisfies a set of recursive equations:

$$\begin{cases} V_\alpha(i;1) = \min\{A(i),\ C(i)+B\}, & 0 \le i \le I \\ V_\alpha(I+1;1) = B & \\ \text{For } n \ge 1, & \\ V_\alpha(i;n+1) = \min\{A(i) + \alpha \sum_{j=0}^{I} p_{ij} V_\alpha(j;n),\ C(i) + B & \\ \qquad + \alpha(qV_\alpha(0;n) + (1-q)\, V_\alpha(I+1;n))\}, & \\ & 0 \le i \le I \\ V_\alpha(I+1;n+1) = B + \alpha(qV_\alpha(0;n) + (1-q)\, V_\alpha(I+1;n)) . & \end{cases} \tag{2.1}$$

Let $V_\alpha(i)$ be the total expected α-discounted cost starting from state i. Then by Theorem 1.1 the existence of a stationary policy

minimizing the total α-discounted cost is guaranteed. Now a special subclass of a set of policies, called a control limit policy, is of interest, which is defined as below:

Definition. A control limit policy is a nonrandomized policy where there is an i for each period n, say i_n, such that for all i with $i < i_n$, the decision at period n is to keep a machine in operation, and for all i with $i \geq i_n$, the decision at period n is to repair it. It is said to be stationary if the i_n's, called the control limits, are constant in n.

A control limit policy has a very simple structure and is intuitively appealing as a rule that an optimal policy must satisfy. Furthermore, if an optimal policy is known to possess this property, the calculation of obtaining the optimal policy can be significantly simplified. But in order to assure that a control limit policy is optimal, certain restrictions must be placed on both the cost structure and the transition probabilities governing deterioration. Investigated here are sufficient conditions under which an optimal policy is assured to be of a control limit form. First consider a finite horizon problem. For any natural number N, we have the following results:

Theorem 2.1. Assume the following conditions hold:

1. $C(i)$ is nondecreasing in i for $0 \leq i \leq I$.
2. $A(i) - C(i)$ is nondecreasing in i for $0 \leq i \leq I$.
3. $P_i(\cdot) \subset P_{i+1}(\cdot)$ for $0 \leq i \leq I-1$.

Then there exists a control limit policy which minimizes the N-stage α-discounted cost of the model.

Proof: We first show that $V_\alpha(i;n)$ is nondecreasing in i for $0 \leq i \leq I$ for each $1 \leq n \leq N$. Proof is by mathematical induction. Look at (2.1) and notice that $V_\alpha(i;1)$ is trivially nondecreasing in i since both $A(i)$ and $C(i)$ are nondecreasing in i from 1 and 2. Suppose $V_\alpha(i;n)$ is nondecreasing in i $(0 \leq i \leq I)$ for $1 \leq n \leq N-1$. Then from 3 and by the inductive assumption, $\sum_{j=0}^{I} p_{ij} V_\alpha(j;n)$ is nondecreasing in i using Lemma 1.5, and hence $A(i) + \alpha \sum_{j=0}^{I} p_{ij} V_\alpha(j;n)$ is nondecreasing in i $(0 \leq i \leq I)$. Also from condition 1, $C(i) + B + \alpha(qV_\alpha(0;n) + (1-q)\, V_\alpha(I+1;n))$ is nondecreasing in i $(0 \leq i \leq I)$. Therefore $V_\alpha(i;n+1)$ is nondecreasing in i $(0 \leq i \leq I)$ since the minimum of two nondecreasing functions is also nondecreasing, completing the mathematical induction.

Consider an n-stage problem $(1 \leq n \leq N)$. For $0 \leq i \leq I$, let

$$f_n(i) = \begin{pmatrix}\text{expected cost of not repair}\\ \text{for a machine in state i at}\\ \text{the beginning followed by}\\ \text{the best rule}\end{pmatrix} - \begin{pmatrix}\text{expected cost of repair}\\ \text{for a machine in state i}\\ \text{at the beginning followed}\\ \text{by the best rule}\end{pmatrix}$$

$$= \{A(i) + \alpha \sum_{j=0}^{I} p_{ij} V_\alpha(j;n-1)\} - \{C(i) + B + \alpha(qV_\alpha(0;n-1) + (1-q)\, V_\alpha(I+1;n-1))\} .$$

Note that the above expression also holds for $n = 1$ by setting $V_\alpha(j;0) = 0$ for any j. $A(i) - C(i)$ is nondecreasing in i from 2.

From 3, $\sum_{j=0}^{I} p_{ij} V_\alpha(j;n-1)$ is nondecreasing in i $(0 \leq i \leq I)$ using Lemma 1.5 since $V_\alpha(j;n-1)$ is nondecreasing in j $(0 \leq j \leq I)$. Other terms are independent of i. Hence $f_n(i)$ is nondecreasing in i $(0 \leq i \leq I)$ for any $1 \leq n \leq N$. Let i_n^* be the smallest $i_n \geq 0$ such that $f_n(i_n) \geq 0$. Note that by the assumption that $A(I)$ is quite large, $f_n(I) > 0$, and hence, $0 \leq i_n^* \leq I$. So for each n $(1 \leq n \leq N)$, there exists i_n^* such that

$$A(i) + \alpha \sum_{j=0}^{I} p_{ij} V_\alpha(j;n-1) < C(i) + B + \alpha(qV_\alpha(0;n-1) + (1-q)V_\alpha(I+1;n-1))$$

if and only if $i < i_n^*$. This implies that at the start of each n-stage problem it is optimal to keep the machine in operation if its state i is less than i_n^*, and it is optimal to repair it if i is greater than or equal to i_n^*, which is a control limit policy. □

Condition 1 says that the material cost of repairing a machine must increase with its degree of deterioration, while 2 means the operating cost must increase more than the increase of the material cost. 3 assures the IFR property on the underlining Markov chain.

The above result can be easily extended to the total expected α-discounted cost case.

<u>Theorem 2.2</u>. Assume the following conditions hold:

1. $C(i)$ is nondecreasing in i for $0 \leq i \leq I$.
2. $A(i) - C(i)$ is nondecreasing in i for $0 \leq i \leq I$.
3. $P_i(\cdot) \subset P_{i+1}(\cdot)$ for $0 \leq i \leq I-1$.

Then there exists a stationary control limit policy R_{i_α} minimizing the total expected α-discounted cost of the model where i_α is its control limit.

Proof: Theorem 1.1 shows that $V_\alpha(i)$ is the unique solution to

$$V_\alpha(i) = \min\{A(i) + \alpha \sum_{j=0}^{I} p_{ij}V_\alpha(j),\ C(i) + B + \alpha(qV_\alpha(0) + (1-q)V_\alpha(I+1))\},$$

$$0 \leq i \leq I \qquad (2.2)$$

$$V_\alpha(I+1) = B + \alpha(qV_\alpha(0) + (1-q)V_\alpha(I+1)) .$$

For $0 \leq i \leq I$, let

$$f(i) = A(i) + \alpha \sum_{j=0}^{I} p_{ij}V_\alpha(j) - \{C(i) + B + \alpha(qV_\alpha(0) + (1-q)V_\alpha(I+1))\}$$

Then Theorem 1.2 guarantees that

$$f(i) = \lim_{n\to\infty} f_n(i) , \qquad 0 \leq i \leq I .$$

Now $f(i)$ is nondecreasing in i $(0 \leq i \leq I)$ since $f_n(i)$ is nondecreasing in i for each $n \geq 1$, which has been proved in the last theorem. Let i_α be the smallest $i \geq 0$ such that $f(i) \geq 0$. Since $f(I) > 0$ from the assumption that $A(I)$ is very large, $0 \leq i_\alpha \leq I$. Then for $0 \leq i < i_\alpha$,

$$A(i) + \alpha \sum_{j=0}^{I} p_{ij} V_\alpha(j) < C(i) + B + \alpha(qV_\alpha(0) + (1-q)V_\alpha(I+1)),$$

and for $i_\alpha \leq i \leq I$,

$$A(i) + \alpha \sum_{j=0}^{I} p_{ij} V_\alpha(j) \geq C(i) + B + \alpha(qV_\alpha(0) + (1-q)V_\alpha(I+1)) .$$

By the uniqueness of the solution to (2.2), a stationary control limit policy R_{i_α} is optimal, where the repair decision is selected if and only if $i \geq i_\alpha$. □

Now consider the long-run average cost case. First a sufficient condition under which Assumption 2 is guaranteed is investigated.

Definition. A machine is said to eventually fail if there exists $n \geq 0$ such that $p_{iI}^{(n)} > 0$ for any i $(0 \leq i \leq I-1)$ where $p_{ij}^{(n)}$ is the probability of n-step transition from state i to state j.

Suppose a machine eventually fails. Then it is clear that no matter which stationary policy is employed, the best condition, state 0, is accessible from any other state since a failed machine must be repaired, and after the repair, it regains the best condition. Therefore the eventual failure property assures Assumption 2. The following theorem then holds.

Theorem 2.3. Assume the following conditions hold:

1. $C(i)$ is nondecreasing in i for $0 \leq i \leq I$.

2. $A(i) - C(i)$ is nondecreasing in i for $0 \leq i \leq I$.

3. $P_i(\cdot) \subset P_{i+1}(\cdot)$ for $0 \leq i \leq I-1$.

Also assume that a machine in the system eventually fails. Then there exists a stationary control limit policy R_{i*} minimizing the long-run expected average cost of the model where i^* is its control limit.

Proof: For each α $(0 < \alpha < 1)$, there exists a stationary control limit policy R_{i_α} minimizing the total α-discounted cost since Theorem 2.2 holds. Choose a sequence $\{\alpha_k\}$ $(k = 0,1,\ldots)$ such that $\lim_{k\to\infty} \alpha_k = 1$ and such that for any α_k, the same control limit policy R_{i*} is optimal. It is possible to have this sequence since the number of stationary control limit rules is finite. Now

$$V_{R_{i*},\alpha_k}(j) \leq V_{R,\alpha_k}(j) \qquad \text{for any } R,\ k \text{ and } j.$$

Hence,

$$(1-\alpha_k)V_{R_{i*},\alpha_k}(j) \leq (1-\alpha_k)V_{R,\alpha_k}(j) \quad \text{for any } R,\ k \text{ and } j.$$

But since Theorem 1.4 holds because of the eventual failure property,

$$V_R(j) = V_R(0) = \lim_{\alpha\to 1}(1-\alpha)V_{R,\alpha}(0), \quad 0 \leq j \leq I.$$

Therefore, for any R and j,

$$\begin{aligned} V_{R_{i*}}(j) = V_{R_{i*}}(0) &= \lim_{k\to\infty}(1-\alpha_k)V_{R_{i*},\alpha_k}(0) \\ &\leq \lim_{k\to\infty}(1-\alpha_k)V_{R,\alpha_k}(0) = V_R(0) = V_R(j), \end{aligned}$$

Therefore the stationary control limit policy R_{i*} minimizes the long-run average cost, where $i*$ is its control limit. □

Conditions 1, 2 and 3 of the previous theorems are known to be sufficient conditions for the optimality of control limit policies for similar systems where replacement takes the place of repair. That means that, as far as sufficient conditions are concerned, the model treated in this section and the corresponding replacement model are essentially the same.

2.2. Special Case with Arbitrary Repair Time Distribution

We now investigate the same model that was treated in the last section with the exception that the class of repair time distributions is now expanded from that of geometric distributions to include any discrete time repair distribution. Let T_i be the repair time of a machine in state i. Assume

$$P\{T_i = k\} = p_k \geq 0, \qquad k = 1,2,\ldots;\ 0 \leq i \leq I$$

where

$$\sum_{k=1}^{\infty} p_k = 1.$$

Note that the repair time distribution is still independent of the state of the machine.

In order to describe the model, a countable number of states labeled $I+1, I+2, \ldots$ is introduced to distinguish the length of repair times. State $I+k$ $(k = 1,2,\ldots)$ is visited when a machine is not completely repaired after k time periods from the start of its repair work. A repaired machine returns to state 0 as before. Let q_k be the probability that a machine is completely repaired in the k-th period after the start of its repair work, given it has not been completely repaired by the end of the (k-1)-th period. That is,

$$q_k = P\{T_i = k | T_i > k-1\} = \frac{p_k}{1 - \sum_{m=1}^{k-1} p_m} ,$$

$$k = 1,2,\ldots; \; 0 \leq i \leq I .$$

Let $B(k)$ $(k = 1,2,\ldots)$ be the labor cost during the k-th period after the beginning of repairing.

As before, $V_\alpha(i;n)$ is the minimum n period α-discounted cost starting from state i, while $V_\alpha(i)$ is the minimum total α-discounted cost starting from state i. Then recursively,

$$
\begin{cases}
V_\alpha(i;1) = \min\{A(i),\ C(i)+B\}, & i = 0,1,\ldots,I \\
V_\alpha(I+k;1) = B(k)\ , & k = 1,2,\ldots \\
\text{For } n \geq 1, & \\
V_\alpha(i;n+1) = \min\{A(i) + \alpha \sum_{j=0}^{I} p_{ij} V_\alpha(j;n),\ C(i) + B + \alpha(q_1 V_\alpha(0;n) & \\
\qquad + (1-q_1)V_\alpha(I+1;n))\}, & i = 0,1,\ldots,I \\
V_\alpha(I+k;n+1) = B(k) + \alpha(q_k V_\alpha(0;n) + (1-q_k)V_\alpha(I+k+1;n)), & \\
 & k = 1,2,\ldots
\end{cases}
\tag{2.3}
$$

We suppose that there exists a finite number M such that $|B(k)| < M$ for any $k \geq 1$. Then all the costs are bounded. Now if the costs are bounded, Theorems 1.1 through 1.4 are known to hold even for the case where the state space is countable rather than finite. Further, compare the third equation in (2.3) with the third equation in (2.1). Notice that they are essentially the same. Therefore the same method as is used in the previous section can be applied here to show that $f(i)$ is nondecreasing in i for $0 \leq i \leq I$, where

$$
f(i) = A(i) + \alpha \sum_{j=0}^{I} p_{ij} V_\alpha(j) - (C(i) + B + \alpha(q_1 V_\alpha(0) + (1-q_1)V_\alpha(I+1)))\ .
$$

As a conclusion we have the following theorem:

Theorem 2.4. Assume the following conditions hold:

1. $C(i)$ is nondecreasing in i for $0 \leq i \leq I$.

2. $A(i) - C(i)$ is nondecreasing in i for $0 \leq i \leq I$.

3. $P_i(\cdot) \subset P_{i+1}(\cdot)$ for $0 \leq i \leq I-1$.

Then there exists a stationary control limit policy R_{i_α} minimizing the total α-discounted cost of the model with arbitrary repair time distribution. Furthermore, if a machine in the system eventually fails, there exists a stationary control limit policy R_{i*} minimizing the long-run average cost.

2.3. General Case

We return to the model where the repair time has a geometric distribution. In Section 2.1 the repair time distribution is independent of the state of a machine to be repaired. In this section the case where the repair time distribution may depend on the machine's state is treated. Namely, for $0 \leq i \leq I$,

$$P\{T_i = j\} = q_i(1-q_i)^{j-1}, \qquad j = 1,2,\ldots$$

The other generalization is the dependence of the labor cost on the machine's state. Define

$B(i)$: labor cost for repairing a machine in state i per period.

The rest of the model description is exactly the same as that in Section 2.1.

We first consider the total α-discounted cost case and see under what conditions some control limit policy optimizes the model. In order to solve this problem, we augment the state space to $0,1,\ldots,I$ and $I+1(0), I+1(1), \ldots, I+1(I)$ so that the state of the machine entering the repair facility can be memorized during its repair period. That is, if a machine in state i is chosen to be repaired, it goes to the repair facility reserved for the state $I+1(i)$, and stays there until its repair is completed. The number of time periods it is in state $I+1(i)$ has the geometric distribution with parameter q_i, and the labor cost $B(i)$ is charged per period while it is in state $I+1(i)$.

This problem again can be formulated as a dynamic programming problem by introducing $V_\alpha(i;n)$ as being the minimum n period expected α-discounted cost starting from state i ($i = 0,1,\ldots,I, I+1(0),\ldots, I+1(I)$).

$$\begin{cases} V_\alpha(i;1) = \min\{A(i), C(i)+B(i)\}, & 0 \le i \le I \\ V_\alpha(I+1(i);1) = B(i), & 0 \le i \le I \\ \text{For } n \ge 1, & \\ V_\alpha(i;n+1) = \min\{A(i) + \alpha \sum_{j=0}^{I} p_{ij} V_\alpha(j;n),\ C(i) + B(i) & \\ \qquad + \alpha(q_i V_\alpha(0;n) + (1-q_i)V_\alpha(I+1(i);n))\}, & \\ & 0 \le i \le I \\ V_\alpha(I+1(i),n+1) = B(i) + \alpha(q_i V_\alpha(0;n) + (1-q_i)V_\alpha(I+1(i);n)), & \\ & 0 \le i \le I . \end{cases} \tag{2.4}$$

The existence of a stationary policy minimizing the total α-discounted cost is apparent since the system is a Markov decision process with discount factor $0 \leq \alpha < 1$. The question is whether or not this optimal policy can be a control limit policy. What makes the analysis more difficult than that previously considered is the fact that the right-hand side of (2.4) depends heavily on state i. The following lemma is useful to derive the conditions under which a control limit policy minimizes the total α-discounted cost.

<u>Lemma 2.5</u>. Assume the following conditions hold:

1. $C(i)$ is nondecreasing in i for $0 \leq i \leq I$.
2. $A(i)$ is nondecreasing in i for $0 \leq i \leq I$.
3. $B(i)$ is nondecreasing in i for $0 \leq i \leq I$.
4. $q_i \geq q_{i+1} > 0$ for $0 \leq i \leq I-1$.
5. $P_i(\cdot) \subset P_{i+1}(\cdot)$ for $0 \leq i \leq I-1$.
6. $C(0) = 0$.

Then,

(a) $V_\alpha(i;n)$ is nondecreasing in i $(0 \leq i \leq I)$ for each $n \geq 1$.

(b) $V_\alpha(I+1(i);n)$ is nondecreasing in i $(0 \leq i \leq I)$ for each $n \geq 1$.

<u>Proof</u>: First notice that for any $n \geq 1$, by setting $V_\alpha(i;0) = 0$ for $i = 0,1,\ldots,I,I+1(0),\ldots,I+1(I)$,

$$
\begin{aligned}
&V_\alpha(I+1(0);n) - V_\alpha(0;n) \\
&\quad = B(0) + \alpha(q_0 V_\alpha(0;n-1) + (1-q_0)V_\alpha(I+1(0);n-1)) \\
&\qquad - \min\{A(0) + \alpha \sum_{j=0}^{I} p_{0j} V_\alpha(j;n-1),\ C(0) + B(0) \\
&\qquad + \alpha(q_0 V_\alpha(0;n-1) + (1-q_0)V_\alpha(I+1(0);n-1))\} \\
&\quad \geq B(0) + \alpha(q_0 V_\alpha(0;n-1) + (1-q_0)V_\alpha(I+1(0);n-1)) \\
&\qquad - \{C(0) + B(0) + \alpha(q_0 V_\alpha(0;n-1) + (1-q_0)V_\alpha(I+1(0);n-1))\} \\
&\quad = -C(0) = 0, \text{ by } 6.
\end{aligned}
$$

The proof is now completed by mathematical induction applied simultaneously on the two statements (a) and (b). Both (a) and (b) obviously hold for $n = 1$, from (2.4) and conditions 1, 2 and 3. Suppose they hold for $n = m \geq 1$. Then for $0 \leq i \leq I-1$,

$$
\begin{aligned}
&q_{i+1}V_\alpha(0;m) + (1-q_{i+1})V_\alpha(I+1(i+1);m) - (q_i V_\alpha(0;m) + (1-q_i)V_\alpha(I+1(i);m)) \\
&\quad = q_{i+1}V_\alpha(0;m) + (1-q_{i+1})(V_\alpha(0;m) + \delta^1_{i,m} + \delta^2_{i,m}) \\
&\qquad - (q_i V_\alpha(0;m) + (1-q_i)(V_\alpha(0;m) + \delta^1_{i,m})) \\
&\quad = (q_i - q_{i+1})\delta^1_{i,m} + (1-q_{i+1})\delta^2_{i,m} \geq 0, \text{ by } 4,
\end{aligned}
$$

where

$$
\begin{aligned}
\delta^1_{i,m} &= V_\alpha(I+1(i);m) - V_\alpha(0;m) \\
&\geq V_\alpha(I+1(0);m) - V_\alpha(0;m) && \text{by inductive assumption} \\
&\geq 0 && \text{by the first note in the proof}
\end{aligned}
$$

and

$$\delta^2_{i,m} = V_\alpha(I+1(i+1);m) - V_\alpha(I+1(i);m) \geq 0 ,$$

by inductive assumption.

Hence $q_i V_\alpha(0;m) + (1-q_i)V_\alpha(I+1(i);m)$ is nondecreasing in i for $0 \leq i \leq I$. Because of this fact, and by 3, $V_\alpha(I+1(i);m+1)$ is nondecreasing in i for $0 \leq i \leq I$. Furthermore, $C(i) + B(i) + \alpha(q_i V_\alpha(0;m) + (1-q_i)V_\alpha(I+1(i);m))$ is also nondecreasing in i if condition 1 is added. Now from 2 and 5 and the induction hypothesis on (a) for $n = m$, $A(i) + \alpha \sum_{j=0}^{I} p_{ij} V_\alpha(j;m)$ is also nondecreasing in i $(0 \leq i \leq I)$ using Lemma 1.5, yielding that $V_\alpha(i;m+1)$ is nondecreasing in i $(0 \leq i \leq I)$. Thus both (a) and (b) hold for $n = m+1$, completing the mathematical induction and the proof of the lemma. □

Interpretation of the conditions in the previous lemma will be given later. Using the results of Lemma 2.5, sufficient conditions for the optimality of a control limit policy can be obtained.

Theorem 2.6. Assume the following conditions hold:

1. $C(i)$ is nondecreasing in i for $0 \leq i \leq I$.
2. $B(i)$ is nondecreasing in i for $0 \leq i \leq I$.
3. $A(i) - (C(i) + B(i)/q_i)$ is nondecreasing in i for $0 \leq i \leq I$.
4. $C(0) = 0$, $A(0) \geq 0$, and $B(0) \geq 0$.
5. $q_i \geq q_{i+1} > 0$ for $0 \leq i \leq I-1$.
6. $P_i(\cdot) \subset P_{i+1}(\cdot)$ for $0 \leq i \leq I-1$.

Then for any $0 \leq \alpha < 1$, there exists a stationary control limit policy minimizing the total expected α-discounted cost of the general model

Proof: First notice that 3 of this theorem implies 2 of Lemma 2.5 with the assistance of conditions 1, 2 and 5. Hence, all the conditions of Lemma 2.5 are assumed here. Also notice that 4 and 1 guarantee that all the costs are nonnegative. Therefore, $V_\alpha(0) \geq 0$, where $V_\alpha(i)$ is the minimum total α-discounted cost starting from state i.

The basic idea to prove this theorem is similar to that of Theorem 2.2, and it is enough to show that by (2.4),

$$f(i) = A(i) + \alpha \sum_{j=0}^{I} p_{ij} V_\alpha(j) - \{C(i)+B(i)+\alpha(q_i V_\alpha(0)+(1-q_i)V_\alpha(I+1(i)))\}$$

is nondecreasing in i for $0 \leq i \leq I$. From Theorem 1.2 and using the result of the previous lemma, $V_\alpha(j;n)$, and hence, $V_\alpha(j)$ are nondecreasing in j for $0 \leq j \leq I$. Therefore by condition 6, $\sum_{j=0}^{I} p_{ij} V_\alpha(j)$ is nondecreasing in i $(0 \leq i \leq I)$ using Lemma 1.5. So it is sufficient to show that

$$g(i) = A(i) - C(i) - B(i) - \alpha(q_i V_\alpha(0) + (1-q_i)V_\alpha(I+1(i)))$$

is nondecreasing in i $(0 \leq i \leq I)$. Now from (2.4) and by Theorem 1.1, $V_\alpha(I+1(i))$ satisfies

$$V_\alpha(I+1(i)) = B(i) + \alpha(q_i V_\alpha(0) + (1-q_i)V_\alpha(I+1(i))), \qquad 0 \leq i \leq I.$$

Therefore

$$V_\alpha(I+1(i)) = \frac{B(i) + \alpha q_i V_\alpha(0)}{1 - \alpha(1-q_i)}, \qquad 0 \leq i \leq I.$$

Substituting this into $g(i)$ yields

$$g(i) = A(i) - C(i) - B(i) - \alpha q_i V_\alpha(0) - \alpha(1-q_i)\ \frac{B(i) + \alpha q_i V_\alpha(0)}{1 - \alpha(1-q_i)}$$

$$= A(i) - C(i) - \frac{B(i)}{1 - \alpha(1-q_i)} - \frac{\alpha q_i V_\alpha(0)}{1 - \alpha(1-q_i)} .$$

Hence,

$$g(i+1) - g(i) - [(A(i+1) - A(i)) - (C(i+1) - C(i))]$$

$$= - \left[\frac{B(i+1)}{1-\alpha(1-q_{i+1})} - \frac{B(i)}{1-\alpha(1-q_i)}\right] - \left[\frac{\alpha q_{i+1} V_\alpha(0)}{1-\alpha(1-q_{i+1})} - \frac{\alpha q_i V_\alpha(0)}{1-\alpha(1-q_i)}\right]$$

$$= - \left[\frac{B(i+1)}{1-\alpha(1-q_{i+1})} - \frac{B(i)}{1-\alpha(1-q_i)}\right] + \frac{(q_i - q_{i+1})\ \alpha(1-\alpha)V_\alpha(0)}{(1-\alpha(1-q_{i+1}))\ (1-\alpha(1-q_i))} .$$

But from 5, and by the fact that $V_\alpha(0) \geq 0$,

$$\frac{(q_i - q_{i+1})(1-\alpha)\alpha V_\alpha(0)}{(1-\alpha(1-q_{i+1}))(1-\alpha(1-q_i))} \geq 0 .$$

Further, if we let $b(i,\alpha)$ be defined as

$$b(i,\alpha) = \frac{B(i+1)}{1-\alpha(1-q_{i+1})} - \frac{B(i)}{1-\alpha(1-q_i)} ,$$

then for $0 \leq \alpha \leq 1$,

$$\frac{\partial b(i,\alpha)}{\partial \alpha} = \frac{(1-q_{i+1})B(i+1)}{(1-\alpha(1-q_{i+1}))^2} - \frac{(1-q_i)\ B(i)}{(1-\alpha(1-q_i))^2} \geq 0$$

because of conditions 2 and 5. Hence, for $0 \leq \alpha < 1$, and for each $0 \leq i \leq I-1$,

$$b(i,\alpha) \leq b(i,1) = \frac{B(i+1)}{q_{i+1}} - \frac{B(i)}{q_i} .$$

Therefore for $0 \le i \le I-1$, by using condition 3,

$$g(i+1) - g(i) \ge (A(i+1) - A(i)) - (C(i+1) - C(i)) - \left(\frac{B(i+1)}{q_{i+1}} - \frac{B(i)}{q_i}\right) \ge 0,$$

which completes the proof. □

Interpretation of each condition in the above theorem is as follows. 1 and 2 indicate that both material and labor costs increase as the condition of the machine to be repaired gets worse. 4 assures the nonnegativity of all costs and that no material cost is incurred if the machine is in the best condition. 5 means that the worse the condition the machine is in, the longer the time required to repair it will be. The IFR property is guaranteed by 6. Now consider condition 3. Note that $B(i)/q_i$ is interpreted as the expected labor cost for a single repair of a machine in state i. Hence 3 says the operating cost must increase more than the increase of the combined material and expected labor costs for repairing a machine.

The long-run average cost version of the previous theorem can be obtained just as in Theorem 2.3, and stated as given below:

Theorem 2.7. Assume all the conditions in Theorem 2.6 hold and furthermore suppose the machine in the system eventually fails. Then there exists a stationary control limit policy minimizing the long-run expected average cost of the general model.

CHAPTER 3

MAINTENANCE MODELS WITH SPARE UNITS

In Chapter 2 several models without spare units are studied. In this chapter the previous models are generalized in the direction to include systems with a finite number of spare units. This generalization seems appropriate since spare units are often introduced in maintenance problems to avoid the system's failure. Obtaining sufficient conditions for the optimality of some kind of *control limit policy* is of interest. Some extensions and modifications are also presented.

3.1. The Model

A general machine repair model with spare units is schematically shown in Fig. 3.1. There is an operating machine and S $(S \geq 1)$ spare machines in the system. The operating machine deteriorates as time goes on and is classified as being in one of $I+1$ states as before. If there is an operating machine, two choices are available by a decision maker at each period: to let it keep operating, or to send it to a repair shop.

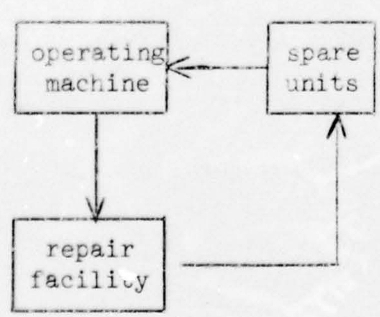

Figure 3.1: A General Machine Repair System With Spare Units.

As before p_{ij} is the one step transition probability from state i to state j when the former decision is selected. When the latter is selected, the operating machine is immediately sent to a repair shop and is replaced by one of the spare machines, if any. The new operating machine begins to operate at the end of the period in its best condition. The repair time T_i of a machine in state i is a random variable having a geometric distribution with parameter q_i, i.e.,

$$P\{T_i = j\} = q_i(1-q_i)^{j-1}, \qquad j = 1,2,\ldots .$$

Machines in the repair shop are independently repaired. If all the machines are in the repair shop, no machine is available and the system fails. In this case there is no option available but to wait until one of the machines is completely repaired. A penalty cost P is assessed per period during the system's failure. Other costs are the operating cost A(i), material cost C(i) and labor cost B(i), which are the same as the ones defined in Chapter 2. The objective is to find a repair policy minimizing the total expected α-discounted cost.

In order to describe the model we need to specify the number of machines in the repair shop coming from the i-th operating condition for each $0 \leq i \leq I$, as well as the state of the operating machine. The state of the system is then represented by

$$X_t = (X_t^0, X_t^1, \ldots, X_t^{I+1}) = (i, s_0, s_1, \ldots, s_I) ,$$

when there is an operating machine at the t-th period whose operating

condition is i, and s_k $(0 \leq k \leq I)$ machines are being repaired which have been in the k-th operating condition when the decision to repair is made. Here,

$$0 \leq i \leq I, \qquad 0 \leq \sum_{k=0}^{I} s_k \leq S, \qquad s_k \geq 0 \quad \text{for} \quad 0 \leq k \leq I.$$

For notational convenience, we write

$$X_t = (0, s_0, s_1, \ldots, s_I)$$

when $\sum_{k=0}^{I} s_k = S + 1$ and $s_k \geq 0$ for $0 \leq k \leq I$, i.e., when all the machines are under repair.

For simplifying the repetitive use of the following sets, we let

$$\mathcal{I} = \{i \mid 0 \leq i \leq I, \ i\text{:integer}\}$$

$$\mathcal{S}^m = \{(s_0, \ldots, s_I) \mid \sum_{k=0}^{I} s_k \leq m, \ s_k \ (0 \leq k \leq I)\text{:nonnegative integer}\}$$

$$\mathcal{S}_0^m = \{(0, s_0, \ldots, s_I) \mid \sum_{k=0}^{I} s_k = m, \ s_k \ (0 \leq k \leq I)\text{:nonnegative integer}\}.$$

This Markov decision model has the following dynamic programming formulation. Let $V_\alpha(i, s_0, \ldots, s_I; n)$ be the minimum expected n period α-discounted cost starting from state $(i, s_0, \ldots, s_I)$. Let $q_{ss'}^{(j)}$ be the probability that s' out of s machines are still under repair at the end of that period given s machines are in the repair shop coming from the j-th operating condition at the beginning of a period. Because of the independence of each machine under repair,

$$q^{(j)}_{s_j s'_j} = \binom{s_j}{s'_j} (1-q_j)^{s'_j} q_j^{s_j - s'_j} , \qquad 0 \le j \le I ,$$

since repair time T_j has the geometric distribution with parameter q_j. Then by setting $V_\alpha(i,s_0,\dots,s_I;0) = 0$ for any feasible $(i,s_0,\dots,s_I)$, $V_\alpha(i,s_0,\dots,s_I;n)$ $(n \ge 1)$ satisfies a set of recursive equations:

$$\begin{cases} V_\alpha(i,s_0,\dots,s_I;n) \\ \quad = \min\{A(i) + \sum_{j=0}^{I} B(j)s_j + \alpha \sum_{i'=0}^{I} \sum_{s'_0=0}^{s_0} \cdots \sum_{s'_I=0}^{s_I} p_{ii'} \\ \qquad \cdot\, q^{(0)}_{s_0 s'_0} \cdots q^{(I)}_{s_I s'_I} V_\alpha(i',s'_0,\dots,s'_I;n-1), \\ \qquad C(i) + \sum_{j=0}^{I} B(j)s_j + B(i) + \alpha \sum_{s'_0=0}^{s_0} \cdots \sum_{s'_i=0}^{s_i+1} \cdots \sum_{s'_I=0}^{s_I} \\ \qquad \cdot\, q^{(0)}_{s_0 s'_0} \cdots q^{(i)}_{s_i+1, s'_i} \cdots q^{(I)}_{s_I s'_I} V_\alpha(0,s'_0,\dots,s'_I;n-1)\} \\ \text{for } i \in \mathcal{I},\ (s_0,\dots,s_I) \in \mathcal{S}^S \text{ and} \qquad\qquad (3.1) \\ V_\alpha(0,s_0,\dots,s_I;n) \\ \quad = P + \sum_{j=0}^{I} B(j)s_j + \alpha \sum_{s'_0=0}^{s_0} \cdots \sum_{s'_I=0}^{s_I} q^{(0)}_{s_0 s'_0} \cdots q^{(I)}_{s_I s'_I} V_\alpha(0,s'_0,\dots,s'_I;n-1) \end{cases}$$

for $(i,s_0,\dots,s_I) \in \mathcal{S}_0^{S+1}$.

Let

$$R_\alpha(i,s_0,\ldots,s_I;n) = \sum_{s_0'=0}^{s_0} \cdots \sum_{s_I'=0}^{s_I} q_{s_0 s_0'}^{(0)} \cdots q_{s_I s_I'}^{(I)} V_\alpha(i,s_0',\ldots,s_I';n) \ . \tag{3.2}$$

Then (3.1) can be simplified to

$$\begin{aligned} &V_\alpha(i,s_0,\ldots,s_I;n) \\ &\quad = \min\{A(i) + \sum_{j=0}^{I} B(j)s_j + \alpha \sum_{i'=0}^{I} p_{ii'} R_\alpha(i',s_0,\ldots,s_I;n-1), \\ &\qquad\quad C(i) + \sum_{j=0}^{I} B(j)s_j + B(i) + \alpha R_\alpha(0,s_0,\ldots,s_i+1,\ldots,s_I;n-1)\} \\ &\quad \text{for } \ i \in \mathcal{I}, \ (s_0,\ldots,s_I) \in \mathcal{S}^S, \end{aligned} \tag{3.3}$$

$$\begin{aligned} &V_\alpha(0,s_0,\ldots,s_I;n) = P + \sum_{j=0}^{I} B(j)s_j + \alpha R_\alpha(0,s_0,\ldots,s_I;n-1) \\ &\quad \text{for } \ (i,s_0,\ldots,s_I) \in \mathcal{S}_0^{S+1} \ . \end{aligned}$$

A simple control limit policy defined in Chapter 2 can not be expected to be an optimal policy since the state of the system depends on the number of machines in the repair shop as well as the condition of an operating machine. The following type of control limit policies is now a suitable candidate for the class of policies satisfied by an optimal policy.

Definition. An *i control limit policy* is a nonrandomized policy where there is an i for each feasible $s = (s_0,\ldots,s_I)$ and for each period $n \geq 1$, say $i_{s,n}$, called the control limit, such that for all $(i,s_0,\ldots,s_I)$ with $i < i_{s,n}$ the decision at period n is to keep the machine in operation and for all $(i,s_0,\ldots,s_I)$ with $i \geq i_{s,n}$ the decision at period n is to repair it. It is called stationary if $i_{s,n}$ is constant in n for each s.

We now investigate the conditions under which the existence of a stationary i control limit policy minimizing the total α-discounted cost is assured. We first prove several lemmas.

Lemma 3.1. If $V(i) = \min(V_1(i), V_2(i))$ for $0 \leq i \leq I$ and if $V_k(i) - V_k(i-1) \geq M$ for $k = 1,2$ and $1 \leq i \leq I$, then $V(i) - V(i-1) \geq M$, $1 \leq i \leq I$.

Proof: For $1 \leq i \leq I$,

$$V(i) - V(i-1) \geq \min(V_1(i),V_2(i)) - \min(V_1(i) - M,\ V_2(i) - M) = M . \qquad \square$$

Lemma 3.2. If $V(i) = \min(V_1(i),V_2(i))$ for $0 \leq i \leq I$ and if $V_k(i) - V_k(i-1) \leq M$ for $k = 1,2$ and $1 \leq i \leq I$, then $V(i) - V(i-1) \leq M$, $1 \leq i \leq I$.

Proof: For $1 \leq i \leq I$,

$$V(i) - V(i-1) \leq \min(V_1(i),V_2(i)) - \min(V_1(i) - M,\ V_2(i) - M) = M . \qquad \square$$

Lemma 3.3. For $i \in \mathcal{I}$, $(s_0,\ldots,s_I) \in \mathcal{S}^{S-1}$ and for $(i,s_0,\ldots,s_I) \in \mathcal{S}_0^S$,

$$R_\alpha(i,s_0,\ldots,s_k+1,\ldots,s_I;n) - R_\alpha(i,s_0,\ldots,s_k,\ldots,s_I;n)$$

$$= (1-q_k) \sum_{s_0'=0}^{s_0} \cdots \sum_{s_k'=0}^{s_k} \cdots \sum_{s_I'=0}^{s_I} q^{(0)}_{s_0 s_0'} \cdots q^{(I)}_{s_I s_I'}$$

$$\cdot \left(V_\alpha(i,s_0',\ldots,s_k'+1,\ldots,s_I';n) - V_\alpha(i,s_0',\ldots,s_k',\ldots,s_I';n)\right)$$

for $0 \le k \le I$ and for $n \ge 1$.

Proof: For $i \in \mathcal{I}$, $(s_0,\ldots,s_I) \in \mathcal{S}^{S-1}$ and for $(i,s_0,\ldots,s_I) \in \mathcal{S}_0^S$, using (3.2),

$$R_\alpha(i,s_0,\ldots,s_k+1,\ldots,s_I;n) - R_\alpha(i,s_0,\ldots,s_k,\ldots,s_I;n)$$

$$= \sum_{s_0'=0}^{s_0} \cdots \sum_{s_{k-1}'=0}^{s_{k-1}} \sum_{s_{k+1}'=0}^{s_{k+1}} \cdots \sum_{s_I'=0}^{s_I} q^{(0)}_{s_0 s_0'} \cdots q^{(k-1)}_{s_{k-1} s_{k-1}'}\, q^{(k+1)}_{s_{k+1} s_{k+1}'} \cdots q^{(I)}_{s_I s_I'}$$

$$\cdot \left[\sum_{s_k'=0}^{s_k+1} q^{(k)}_{s_k+1, s_k'} V_\alpha(i,s_0',\ldots,s_I';n) - \sum_{s_k'=0}^{s_k} q^{(k)}_{s_k s_k'} V_\alpha(i,s_0',\ldots,s_I';n) \right].$$

Now use the following relation. For nonnegative integers m and n,

$$\binom{m}{n-1} + \binom{m}{n} = \binom{m+1}{n}$$

Note that the above equation holds for $n = 0$ and for $n > m$ by setting

$$\binom{m}{m+k} = 0 \quad \text{for} \quad k \geq 1 \quad \text{and} \quad \binom{m}{0} = 1, \quad \binom{m}{-1} = 0 .$$

Then

$$\sum_{s_k'=0}^{s_k+1} q^{(k)}_{s_k+1,s_k'} V_\alpha(i,s_0',\ldots,s_I';n) - \sum_{s_k'=0}^{s_k} q^{(k)}_{s_k s_k'} V_\alpha(i,s_0',\ldots,s_I';n)$$

$$= \sum_{s_k'=0}^{s_k+1} \binom{s_k+1}{s_k'} (1-q_k)^{s_k'} q_k^{s_k+1-s_k'} V_\alpha(i,s_0',\ldots,s_I';n)$$

$$- \sum_{s_k'=0}^{s_k} \binom{s_k}{s_k'} (1-q_k)^{s_k'} q_k^{s_k-s_k'} V_\alpha(i,s_0',\ldots,s_I';n)$$

$$= \sum_{s_k'=0}^{s_k+1} \binom{s_k}{s_k'-1} (1-q_k)^{s_k'} q_k^{s_k+1-s_k'} V_\alpha(i,s_0',\ldots,s_I';n)$$

$$+ \sum_{s_k'=0}^{s_k+1} \binom{s_k}{s_k'} (1-q_k)^{s_k'} q_k^{s_k+1-s_k'} V_\alpha(i,s_0',\ldots,s_I';n)$$

$$- \sum_{s_k'=0}^{s_k} \binom{s_k}{s_k'} (1-q_k)^{s_k'} q_k^{s_k-s_k'} V_\alpha(i,s_0',\ldots,s_I';n)$$

$$= \sum_{s_k'=0}^{s_k} \binom{s_k}{s_k'} (1-q_k)^{s_k'+1} q_k^{s_k-s_k'} V_\alpha(i,s_0',\ldots,s_k'+1,\ldots,s_I';n)$$

$$+ \sum_{s_k'=0}^{s_k} (q_k-1) \binom{s_k}{s_k'} (1-q_k)^{s_k'} q_k^{s_k-s_k'} V_\alpha(i,s_0',\ldots,s_I';n)$$

$$= (1-q_k) \Bigg[\sum_{s_k'=0}^{s_k} q^{(k)}_{s_k s_k'} V_\alpha(i,s_0',\ldots,s_k'+1,\ldots,s_I';n)$$

$$- \sum_{s_k'=0}^{s_k} q^{(k)}_{s_k s_k'} V_\alpha(i,s_0',\ldots,s_I';n) \Bigg] .$$

Hence,

$$R_\alpha(i,s_0,\ldots,s_k+1,\ldots,s_I;n) - R_\alpha(i,s_0,\ldots,s_k,\ldots,s_I;n)$$

$$= (1-q_k) \sum_{s_0'=0}^{s_0} \cdots \sum_{s_k'=0}^{s_k} \cdots \sum_{s_I'=0}^{s_I} q^{(0)}_{s_0 s_0'} \cdots q^{(I)}_{s_I s_I'}$$

$$\cdot (V_\alpha(i,s_0',\ldots,s_k'+1,\ldots,s_I';n) - V_\alpha(i,s_0',\ldots,s_k',\ldots,s_I';n)). \quad \square$$

<u>Lemma 3.4</u>. Assume the following conditions hold:

1. $0 < q_{k+1} \leq q_k \leq 1$ for $0 \leq k \leq I-1$.

2. $P \geq C(0)$.

3. $0 \leq B(k) \leq B(k+1)$ for $0 \leq k \leq I-1$.

Then the following inequalities hold for each $n \geq 1$, k and for $i \in \mathcal{I}$, $(s_0,\ldots,s_I) \in \mathcal{S}^{S-1}$ and for $(i,s_0,\ldots,s_I) \in \mathcal{S}_0^S$.

(a) $V_\alpha(i,s_0,\ldots,s_k,s_{k+1}+1,\ldots,s_I;n) \geq V_\alpha(i,s_0,\ldots,s_k+1,s_{k+1},\ldots,s_I;n)$.

(b) $R_\alpha(i,s_0,\ldots,s_k,s_{k+1}+1,\ldots,s_I;n) \geq R_\alpha(i,s_0,\ldots,s_k+1,s_{k+1},\ldots,s_I;n)$.

(c) $V_\alpha(i,s_0,\ldots,s_k+1,\ldots,s_I;n) \geq V_\alpha(i,s_0,\ldots,s_k,\ldots,s_I;n)$.

(d) $R_\alpha(i,s_0,\ldots,s_k+1,\ldots,s_I;n) \geq R_\alpha(i,s_0,\ldots,s_k,\ldots,s_I;n)$.

<u>Proof</u>: First notice that (c) implies (d) using Lemma 3.3. Also by the same lemma,

$$R_\alpha(i,s_0,\dots,s_k,s_{k+1}+1,\dots,s_I;n) - R_\alpha(i,s_0,\dots,s_k+1,s_{k+1},\dots,s_I;n)$$

$$= R_\alpha(i,s_0,\dots,s_k,s_{k+1}+1,\dots,s_I;n) - R_\alpha(i,s_0,\dots,s_I;n)$$

$$- (R_\alpha(i,s_0,\dots,s_k+1,\dots,s_I;n) - R_\alpha(i,s_0,\dots,s_I;n))$$

$$= (1-q_{k+1}) \sum_{s_0'=0}^{s_0} \dots \sum_{s_I'=0}^{s_I} q^{(0)}_{s_0 s_0'} \dots q^{(I)}_{s_I s_I'}$$

$$\cdot (V_\alpha(i,s_0',\dots,s_{k+1}'+1,\dots,s_I';n) - V_\alpha(i,s_0',\dots,s_{k+1}',\dots,s_I';n))$$

$$- (1-q_k) \sum_{s_0'=0}^{s_0} \dots \sum_{s_I'=0}^{s_I} q^{(0)}_{s_0 s_0'} \dots q^{(I)}_{s_I s_I'}$$

$$\cdot (V_\alpha(i,s_0',\dots,s_k'+1,\dots,s_I';n) - V_\alpha(i,s_0',\dots,s_k',\dots,s_I';n))$$

$$= (1-q_k) \sum_{s_0'=0}^{s_0} \dots \sum_{s_I'=0}^{s_I} q^{(0)}_{s_0 s_0'} \dots q^{(I)}_{s_I s_I'}$$

$$\cdot (V_\alpha(i,s_0',\dots,s_{k+1}'+1,\dots,s_I';n) - V_\alpha(i,s_0',\dots,s_k'+1,\dots,s_I';n))$$

$$+ (q_k - q_{k+1}) \sum_{s_0'=0}^{s_0} \dots \sum_{s_I'=0}^{s_I} q^{(0)}_{s_0 s_0'} \dots q^{(I)}_{s_I s_I'}$$

$$\cdot (V_\alpha(i,s_0',\dots,s_{k+1}'+1,\dots,s_I';n) - V_\alpha(i,s_0',\dots,s_{k+1}',\dots,s_I',n)).$$

So by condition 1, if (a) and (c) hold then the last expression becomes nonnegative, and hence (b) holds.

What is left is to prove (a) and (c), and this is done by mathematical induction. For $n = 1$, (a) is obvious because of 3 and

(c) is also clear by 2 and 3. Suppose they both hold for $n = m-1 \geq 1$. Then by the previous argument, (a) through (d) hold for $n = m-1$. Consider the case for $n = m$. Take (a) case first. For $(i, s_0, \ldots, s_I) \in \mathcal{S}_0^S$,

$$V_\alpha(0, s_0, \ldots, s_{k+1}+1, \ldots, s_I; m) - V_\alpha(0, s_0, \ldots, s_k+1, \ldots, s_I; m)$$

$$= P + \sum_{j=0}^{I} B(j)s_j + B(k+1) + \alpha R_\alpha(0, s_0, \ldots, s_{k+1}+1, \ldots, s_I; m-1)$$

$$- (P + \sum_{j=0}^{I} B(j)s_j + B(k) + \alpha R_\alpha(0, s_0, \ldots, s_k+1, \ldots, s_I; m-1))$$

$$\geq 0,$$

from 3 and by inductive assumption on (b) for $n = m-1$.

For $i \in \mathcal{I}$, $(s_0, \ldots, s_I) \in \mathcal{S}^{S-1}$, we compare the corresponding values term by term using (3.3).

For notational convenience if

$$V(i) = \min\{V_1(i), V_2(i), \ldots, V_m(i)\},$$

in order to represent the k-th term $V_k(i)$, we write $\langle V(i) \rangle_{k\text{-th}}$ $(1 \leq k \leq m)$ throughout this paper.

$$\langle V_\alpha(i, s_0, \ldots, s_{k+1}+1, \ldots, s_I; m) \rangle_{1\text{-st}} - \langle V_\alpha(i, s_0, \ldots, s_k+1, \ldots, s_I; m) \rangle_{1\text{-st}}$$

$$= A(i) + \sum_{j=0}^{I} B(j)s_j + B(k+1) + \alpha \sum_{i'=0}^{I} p_{ii'} R_\alpha(i', s_0, \ldots, s_{k+1}+1, \ldots, s_I, m-1)$$

$$- (A(i) + \sum_{j=0}^{I} B(j)s_j + B(k) + \alpha \sum_{i'=0}^{I} p_{ii'} R_\alpha(i', s_0, \ldots, s_k+1, \ldots, s_I; m-1))$$

$$\geq 0,$$

from 3 and by inductive assumption on (b) for $n = m-1$. Similarly, we have

$$\langle V_\alpha(i,s_0,\ldots,s_{k+1}+1,\ldots,s_I;m)\rangle_{2\text{-nd}} - \langle V_\alpha(i,s_0,\ldots,s_k+1,\ldots,s_I;m)\rangle_{2\text{-nd}}$$

$$\geq 0.$$

Therefore, by Lemma 3.1, for $i \in \mathscr{I}$, $(s_0,\ldots,s_I) \in \mathscr{S}^{S-1}$,

$$V_\alpha(i,s_0,\ldots,s_{k+1}+1,\ldots,s_I;m) - V_\alpha(i,s_0,\ldots,s_k+1,\ldots,s_I;m) \geq 0 .$$

Hence (a) holds for $n = m$. Consider (c) case. For $i \in \mathscr{I}$, $(s_0,\ldots,s_I) \in \mathscr{S}^{S-1}$, by using the similar technique,

$$\langle V_\alpha(i,s_0,\ldots,s_k+1,\ldots,s_I;m)\rangle_{1\text{-st}} - \langle V_\alpha(i,s_0,\ldots,s_k,\ldots,s_I;m)\rangle_{1\text{-st}}$$

$$= A(i) + \sum_{j=0}^{I} B(j)s_j + B(k) + \alpha \sum_{i'=0}^{I} p_{ii'}R_\alpha(i',s_0,\ldots,s_k+1,\ldots,s_I;m-1)$$

$$- (A(i) + \sum_{j=0}^{I} B(j)s_j + \alpha \sum_{i'=0}^{I} p_{ii'}R_\alpha(i',s_0,\ldots,s_k,\ldots,s_I;m-1))$$

$$= B(k) + \alpha \sum_{i'=0}^{I} p_{ii'}(R_\alpha(i',s_0,\ldots,s_k+1,\ldots,s_I;m-1)$$

$$- R_\alpha(i',s_0,\ldots,s_k,\ldots, s_I;m-1))$$

$$\geq 0,$$

from 3 and by inductive assumption on (d) for $n = m-1$. In a similar manner, we have

$$\langle V_\alpha(i,s_0,\ldots,s_k+1,\ldots,s_I;m)\rangle_{2\text{-nd}} - \langle V_\alpha(i,s_0,\ldots,s_k,\ldots,s_I;m)\rangle_{2\text{-nd}}$$

$$\geq 0$$

which gives us that for $i \in \mathcal{J}$, $(s_0,\ldots,s_I) \in \mathcal{S}^{S-1}$,

$$V_\alpha(i,s_0,\ldots,s_k+1,\ldots,s_I;m) \geq V_\alpha(i,s_0,\ldots,s_k,\ldots,s_I;m).$$

For $(i,s_0,\ldots,s_I) \in \mathcal{S}_0^S$,

$$V_\alpha(0,s_0,\ldots,s_k+1,\ldots,s_I;m) - V_\alpha(0,s_0,\ldots,s_k,\ldots,s_I;m)$$

$$\geq P + \sum_{j=0}^{I} B(j)s_j + B(k) + \alpha R_\alpha(0,s_0,\ldots,s_k+1,\ldots,s_I;m-1)$$

$$- (C(0) + \sum_{j=0}^{I} B(j)s_j + B(0) + \alpha R_\alpha(0,s_0+1,\ldots,s_k,\ldots,s_I;m-1))$$

$$= P - C(0) + B(k) - B(0)$$

$$+ \alpha(R_\alpha(0,s_0,\ldots,s_k+1,\ldots,s_I;m-1)-R_\alpha(0,s_0+1,\ldots,s_k,\ldots,s_I;m-1))$$

$$\geq 0,$$

by 2 and 3 and inductive assumption on (b) for $n = m-1$.

Hence for all cases (c) holds for $n = m$. Thus both (a) and (c) hold for $n = m$, completing the mathematical induction and the proof. □

<u>Lemma 3.5</u>. Assume the following conditions hold:

1. $0 < q_{k+1} \leq q_k \leq 1$ for $0 \leq k \leq I-1$.
2. $P \geq C(0)$.
3. $0 \leq B(k) \leq B(k+1)$ for $0 \leq k \leq I-1$.

4. $P_i(\cdot) \subset P_{i+1}(\cdot)$ for $0 \leq i \leq I-1$ where $P_i(j) = \sum_{k \leq j} p_{ik}$.

5. $A(i)$ is nondecreasing in i for $0 \leq i \leq I$.

6. $C(i)$ is nondecreasing in i for $0 \leq i \leq I$.

Then both $V_\alpha(i,s_0,\ldots,s_I;n)$ and $R_\alpha(i,s_0,\ldots,s_I;n)$ are nondecreasing in i $(0 \leq i \leq I)$ for $(s_0,\ldots,s_I) \in \mathcal{S}^S$ and $n \geq 1$.

Proof: First note that if $V_\alpha(i,s_0,\ldots,s_I;n)$ is nondecreasing in i $(0 \leq i \leq I)$, then by the definition of $R_\alpha(i,s_0,\ldots,s_I;n)$, which is equation (3.2), $R_\alpha(i,s_0,\ldots,s_I;n)$ is also nondecreasing in i $(0 \leq i \leq I)$.

Now the proof is by mathematical induction. For $n = 1$,

$$V_\alpha(i,s_0,\ldots,s_I,1) = \min\{A(i) + \sum_{j=0}^{I} B(j)s_j,\ C(i) + \sum_{j=0}^{I} B(j)s_j + B(i)\}$$

is clearly nondecreasing in i by 3, 5 and 6. Suppose $V_\alpha(i,s_0,\ldots,s_I;m-1)$ $(m \geq 2)$ is nondecreasing in i $(0 \leq i \leq I)$. For $(s_0,\ldots,s_I) \in \mathcal{S}^S$,

$$\langle V_\alpha(i,s_0,\ldots,s_I;m)\rangle_{1\text{-st}}$$
$$= A(i) + \sum_{j=0}^{I} B(j)s_j + \alpha \sum_{i'=0}^{I} p_{ii'}R_\alpha(i',s_0,\ldots,s_I;m-1).$$

As $R_\alpha(i',s_0,\ldots,s_I;m-1)$ is nondecreasing in i' $(0 \leq i' \leq I)$, by 4 and using Lemma 1.5, $\sum_{i'=0}^{I} p_{ii'}R_\alpha(i',s_0,\ldots,s_I;m-1)$ is nondecreasing in i $(0 \leq i \leq I)$. Further $A(i)$ is nondecreasing in i $(0 \leq i \leq I)$, by 5. Hence $\langle V_\alpha(i,s_0,\ldots,s_I;m)\rangle_{1\text{-st}}$ is nondecreasing in i $(0 \leq i \leq I)$. Also,

$$\langle V_\alpha(i,s_0,\dots,s_I;m)\rangle_{2\text{-nd}}$$
$$= C(i) + \sum_{j=0}^{I} B(j)s_j + B(i) + \alpha R_\alpha(0,s_0,\dots,s_i+1,\dots,s_I;m-1) .$$

Notice that Lemma 3.4 holds because of conditions 1, 2 and 3. So $R_\alpha(0,s_0,\dots,s_i+1,\dots,s_I;m-1)$ is nondecreasing in i $(0 \le i \le I)$ by (b) of Lemma 3.4. Therefore $\langle V_\alpha(i,s_0,\dots,s_I;m)\rangle_{2\text{-nd}}$ is also nondecreasing in i $(0 \le i \le I)$ since both $C(i)$ and $B(i)$ are nondecreasing in i by 3 and 6. Thus $V_\alpha(i,s_0,\dots,s_I;m)$ is nondecreasing in i $(0 \le i \le I)$, completing the induction and the lemma. □

The property that $V_\alpha(i,s_0,\dots,s_I;n)$ is nondecreasing in i $(0 \le i \le I)$ is necessary but monotonicity is not enough to show that an optimal policy is assured to be of an i control limit form. In order to obtain other conditions, bounds on $V_\alpha(i,s_0,\dots,s_k+1,\dots,s_I,n)$ $- V_\alpha(i,s_0,\dots,s_k,\dots,s_I;n)$ and $V_\alpha(i,s_0,\dots,s_k,s_{k+1}+1,\dots,s_I;n)$ $- V_\alpha(i, s_0,\dots,s_k+1,s_{k+1},\dots,s_I;n)$ must be calculated. Lower bounds on them are given in Lemma 3.4. Upper bounds are shown in the next two lemmas.

<u>Lemma 3.6</u>. Assume conditions 1, 3, 4, 5 and 6 of Lemma 3.5 hold and furthermore assume the following condition holds:

2. $P \ge C(0) + B(0)$.

Then for $0 \le k \le I$ and $n \ge 1$,

$$V_\alpha(i,s_0,\ldots,s_k+1,\ldots,s_I;n) - V_\alpha(i,s_0,\ldots,s_k,\ldots,s_I;n) \leq U^k_{\alpha,n}$$

for $i \in \mathscr{I}$, $(s_0,\ldots,s_I) \in \mathscr{S}^{S-1}$ and for $(i,s_0,\ldots,s_I) \in \mathscr{S}^S_0$, where

$$U^k_{\alpha,n} = \frac{(1-\alpha^n(1-q_k)^n)G(k)}{1-\alpha(1-q_k)}$$

and

$$G(k) = P + B(k) - \min\{A(0),\ C(0) + B(0)\} .$$

<u>Proof</u>: Mathematical induction is again used. For $n = 1$, if $i \in \mathscr{I}$ and if $(s_0,\ldots,s_I) \in \mathscr{S}^{S-1}$, then

$$V_\alpha(i,s_0,\ldots,s_k+1,\ldots,s_I;1) - V_\alpha(i,s_0,\ldots,s_k,\ldots,s_I;1) = B(k) .$$

For $(i,s_0,\ldots,s_I) \in \mathscr{S}^S_0$,

$$V_\alpha(0,s_0,\ldots,s_k+1,\ldots,s_I;1) - V_\alpha(0,s_0,\ldots,s_k,\ldots,s_I;1)$$

$$= P + \sum_{j=0}^{I} B(j)s_j + B(k) - \min\{A(0) + \sum_{j=0}^{I} B(j)s_j, C(0) + \sum_{j=0}^{I} B(j)s_j + B(0)\}$$

$$= P + B(k) - \min\{A(0),\ C(0) + B(0)\} = G(k).$$

Now,

$$G(k) = P + B(k) - \min\{A(0),\ C(0) + B(0)\}$$

$$\geq P + B(k) - (C(0) + B(0)) \geq B(k), \text{ by 2.}$$

Hence,

$$V_\alpha(i,s_0,\ldots,s_k+1,\ldots,s_I;1) - V_\alpha(i,s_0,\ldots,s_k,\ldots,s_I;1) \leq G(k) = U^k_{\alpha,1}.$$

So the hypothesis is true for $n = 1$. Suppose it is true for $n = m-1 \geq 1$, and consider the case for $n = m$. Now from Lemma 3.3,

$$R_\alpha(i,s_0,\ldots,s_k+1,\ldots,s_I;m-1) - R_\alpha(i,s_0,\ldots,s_k,\ldots,s_I;m-1)$$

$$= (1-q_k) \sum_{s_0'=0}^{s_0} \cdots \sum_{s_I'=0}^{s_I} q^{(0)}_{s_0 s_0'} \cdots q^{(I)}_{s_I s_I'}$$

$$\cdot \, (V_\alpha(i,s_0,\ldots,s_k+1,\ldots,s_I;m-1) - V_\alpha(i,s_0,\ldots,s_k,\ldots, s_I,m-1))$$

$$\leq (1-q_k)\, U^k_{\alpha,m-1} \text{ , by inductive assumption.}$$

For $(s_0,\ldots,s_I) \in \mathcal{S}^{S-1}$,

$$\langle V_\alpha(i,s_0,\ldots,s_k+1,\ldots,s_I;m)\rangle_{1\text{-st}} - \langle V_\alpha(i,s_0,\ldots,s_k,\ldots,s_I;m)\rangle_{1\text{-st}}$$

$$= A(i) + \sum_{j=0}^{I} B(j)s_j + B(k) + \alpha \sum_{i'=0}^{I} p_{ii'} R_\alpha(i',s_0,\ldots,s_k+1,\ldots,s_I;m-1)$$

$$- \left(A(i) + \sum_{j=0}^{I} B(j)s_j + \alpha \sum_{i'=0}^{I} p_{ii'} R_\alpha(i',s_0,\ldots,s_k,\ldots,s_I;m-1)\right)$$

$$\leq B(k) + \alpha \sum_{i'=0}^{I} p_{ii'}(1-q_k)U^k_{\alpha,m-1} = B(k) + \alpha(1-q_k)U^k_{\alpha,m-1} .$$

In a similar manner, we can get the same upper bound on the difference of the corresponding second terms. Hence for $(s_0,\ldots,s_I) \in \mathcal{S}^{S-1}$, $i \in \mathcal{I}$, using Lemma 3.2,

$$V_\alpha(i,s_0,\ldots,s_k+1,\ldots,s_I;m) - V_\alpha(i,s_0,\ldots,s_k,\ldots,s_I;m)$$

$$\leq B(k) + \alpha(1-q_k)U^k_{\alpha,m-1}.$$

For $(i,s_0,\ldots,s_I) \in \mathscr{S}^S_0$,

$$V_\alpha(0,s_0,\ldots,s_k+1,\ldots,s_I;m) - V_\alpha(0,s_0,\ldots,s_k,\ldots,s_I;m)$$

$$= P + \sum_{j=0}^{I} B(j)s_j + B(k) + \alpha R_\alpha(0,s_0,\ldots,s_k+1,\ldots,s_I;m-1)$$

$$- \min\{A(0) + \sum_{j=0}^{I} B(j)s_j + \alpha \sum_{i'=0}^{I} p_{0i'}R_\alpha(i',s_0,\ldots,s_I;m-1),$$

$$C(0) + \sum_{j=0}^{I} B(j)s_j + B(0) + \alpha R_\alpha(0,s_0+1,\ldots,s_I;m-1)\}$$

$$\leq P + B(k) + \alpha R_\alpha(0,s_0,\ldots,s_k+1,\ldots,s_I;m-1)$$

$$- \min\{A(0) + \alpha R_\alpha(0,s_0,\ldots,s_I;m-1),$$

$$C(0) + B(0) + \alpha R_\alpha(0,s_0,\ldots,s_I;m-1)\}$$ by Lemmas 3.4 and 3.5

$$= G(k) + \alpha(R_\alpha(0,s_0,\ldots,s_k+1,\ldots,s_I;m-1) - R_\alpha(0,s_0,\ldots,s_I;m-1))$$

$$\leq G(k) + \alpha(1-q_k)U^k_{\alpha,m-1},$$ by inductive assumption.

As $G(k) \geq B(k)$ by 2, for all cases,

$$V_\alpha(i,s_0,\ldots,s_k+1,\ldots,s_I;m) - V_\alpha(i,s_0,\ldots,s_k,\ldots,s_I;m)$$

$$\leq G(k) + \alpha(1-q_k)U^k_{\alpha,m-1} = U^k_{\alpha,m}$$

which completes the induction and the proof of the lemma. □

Let

$$U_\alpha^k = \lim_{n \to \infty} U_{\alpha,n}^k = \frac{G(k)}{1 - \alpha(1-q_k)} .$$

Then as $U_{\alpha,n}^k$ is nondecreasing in n for any α and k, under the same conditions as in Lemma 3.6, we have

$$V_\alpha(i,s_0,\ldots,s_k+1,\ldots,s_I;n) - V_\alpha(i,s_0,\ldots,s_k, \ldots ,s_I;n) \leq U_\alpha^k .$$

Using the above lemma, we can derive an upper bound on $V_\alpha(i,s_0,\ldots,s_{k+1}+1,\ldots,s_I;n) - V_\alpha(i,s_0,\ldots, s_k+1,\ldots,s_I;n)$ which is in turn used to obtain the main theorem.

Lemma 3.7. If all the conditions in Lemma 3.6 hold, then for $i \in \mathcal{I}$, $(s_0,\ldots,s_1) \in \mathcal{S}^{S-1}$ and for $(i,s_0,\ldots,s_I) \in \mathcal{S}_0^S$,

$$V_\alpha(i,s_0,\ldots,s_k,s_{k+1}+1,\ldots,s_I;n) - V_\alpha(i,s_0,\ldots,s_k+1,s_{k+1},\ldots,s_I;n)$$

$$\leq M_{\alpha,n}^k \leq M_\alpha^k , \qquad 0 \leq k \leq I-1,\ n \geq 1 ,$$

where

$$M_{\alpha,n}^k = \frac{1-\alpha^n(1-q_k)^n}{1-\alpha(1-q_k)} (B(k+1) - B(k) + \alpha(q_k-q_{k+1})U_\alpha^{k+1})$$

and

$$M_\alpha^k = \lim_{n \to \infty} M_{\alpha,n}^k = \frac{B(k+1) - B(k) + \alpha(q_k-q_{k+1})U_\alpha^{k+1}}{1-\alpha(1-q_k)} .$$

$\underline{\text{Proof}}$: Mathematical induction is applied. For $n = 1$, if $(i, s_0, \ldots, s_I) \in \mathcal{S}_0^S$,

$$V_\alpha(0, s_0, \ldots, s_k, s_{k+1}+1, \ldots, s_I; 1) - V_\alpha(0, s_0, \ldots, s_k+1, s_{k+1}, \ldots, s_I; 1)$$

$$= P + \sum_{j=0}^{I} B(j)s_j + B(k+1) - (P + \sum_{j=0}^{I} B(j)s_j + B(k))$$

$$= B(k+1) - B(k)$$

$$\leq B(k+1) - B(k) + \alpha(q_k - q_{k+1})U_\alpha^{k+1} = M_{\alpha,1}^k \quad \text{by 1.}$$

Similarly for $i \in \mathcal{I}$ and $(s_0, \ldots, s_I) \in \mathcal{S}^{S-1}$,

$$V_\alpha(i, s_0, \ldots, s_k, s_{k+1}+1, \ldots, s_I; 1) - V_\alpha(i, s_0, \ldots, s_k+1, s_{k+1}, \ldots, s_I; 1)$$

$$= B(k+1) - B(k) \leq M_{\alpha,1}^k .$$

So the hypothesis holds for $n = 1$. Suppose that it holds for $n = m-1 \geq 1$. Then

$$R_\alpha(i, s_0, \ldots, s_{k+1}+1, \ldots, s_I; m-1) - R_\alpha(i, s_0, \ldots, s_k+1, \ldots, s_I; m-1)$$

$$= (R_\alpha(i, s_0, \ldots, s_{k+1}+1, \ldots, s_I; m-1) - R_\alpha(i, s_0, \ldots, s_I; m-1))$$

$$- (R_\alpha(i, s_0, \ldots, s_k+1, \ldots, s_I; m-1) - R_\alpha(i, s_0, \ldots, s_I; m-1))$$

$$= (1-q_{k+1}) \sum_{s_0'=0}^{s_0} \cdots \sum_{s_I'=0}^{s_I} q^{(0)}_{s_0 s_0'} \cdots q^{(I)}_{s_I s_I'}$$

$$\cdot (V_\alpha(i,s_0',\ldots,s_{k+1}'+1,\ldots,s_I';m-1) - V_\alpha(i,s_0',\ldots,s_{k+1}',\ldots,s_I';m-1))$$

$$- (1-q_k) \sum_{s_0'=0}^{s_0} \cdots \sum_{s_I'=0}^{s_I} q^{(0)}_{s_0 s_0'} \cdots q^{(I)}_{s_I s_I'}$$

$$\cdot (V_\alpha(i,s_0',\ldots,s_k'+1,\ldots,s_I';m-1) - V_\alpha(i,s_0',\ldots,s_k',\ldots,s_I';m-1)),$$

by Lemma 3.3

$$= (1-q_k) \sum_{s_0'=0}^{s_0} \cdots \sum_{s_I'=0}^{s_I} q^{(0)}_{s_0 s_0'} \cdots q^{(I)}_{s_I s_I'}$$

$$\cdot (V_\alpha(i,s_0',\ldots,s_{k+1}'+1,\ldots,s_I';m-1) - V_\alpha(i,s_0',\ldots,s_k'+1,\ldots,s_I';m-1))$$

$$+ (q_k-q_{k+1}) \sum_{s_0'=0}^{s_0} \cdots \sum_{s_I'=0}^{s_I} q^{(0)}_{s_0 s_0'} \cdots q^{(I)}_{s_I s_I'}$$

$$\cdot (V_\alpha(i,s_0',\ldots,s_{k+1}'+1,\ldots,s_I';m-1) - V_\alpha(i,s_0',\ldots,s_{k+1}',\ldots,s_I';m-1))$$

$$\leq (1-q_k) \sum_{s_0'=0}^{s_0} \cdots \sum_{s_0'=0}^{s_I} q^{(0)}_{s_0 s_0'} \cdots q^{(I)}_{s_I s_I'} M^k_{\alpha,m-1}$$

$$+ (q_k-q_{k+1}) \sum_{s_0'=0}^{s_0} \cdots \sum_{s_I'}^{s_I} q^{(0)}_{s_0 s_0'} \cdots q^{(I)}_{s_I s_I'} U^{k+1}_\alpha ,$$

by Lemma 3.6 and inductive assumption

$$= (1-q_k)M^k_{\alpha,m-1} + (q_k-q_{k+1})U^{k+1}_\alpha .$$

Therefore, for $(i,s_0,\ldots,s_I) \in \mathcal{S}_0^S$,

$$V_\alpha(0,s_0,\ldots,s_{k+1}+1,\ldots,s_I;m) - V_\alpha(0,s_0,\ldots,s_k+1,\ldots,s_I;m)$$

$$= P + \sum_{j=0}^{I} B(j)s_j + B(k+1) + \alpha R_\alpha(0,s_0,\ldots,s_{k+1}+1,\ldots,s_I;m-1)$$

$$- (P + \sum_{j=0}^{I} B(j)s_j + B(k) + \alpha R_\alpha(0,s_0,\ldots,s_k+1,\ldots,s_I;m-1))$$

$$\leq B(k+1) - B(k) + \alpha(q_k - q_{k+1})U_\alpha^{k+1} + \alpha(1-q_k)M_{\alpha,m-1}^k .$$

For $i \in \mathcal{I}$ and $(s_0,\ldots,s_I) \in \mathcal{S}^{S-1}$, by comparing $V_\alpha(i,s_0,\ldots,s_{k+1}+1,\ldots,s_I;m)$ and $V_\alpha(i,s_0,\ldots,s_k+1,\ldots,s_I;m)$ term by term and using Lemma 3.2, we can similarly show that

$$V_\alpha(i,s_0,\ldots,s_{k+1}+1,\ldots,s_I;m) - V_\alpha(i,s_0,\ldots,s_k+1,\ldots,s_I;m)$$

$$\leq B(k+1) - B(k) + \alpha(q_k - q_{k+1})U_\alpha^{k+1} + \alpha(1-q_k)M_{\alpha,m-1}^k .$$

Hence, for all cases,

$$V_\alpha(i,s_0,\ldots,s_{k+1}+1,\ldots,s_I;m) - V_\alpha(i,s_0,\ldots,s_k+1,\ldots,s_I;m)$$

$$\leq B(k+1) - B(k) + \alpha(q_k - q_{k+1})U_\alpha^{k+1} + \alpha(1-q_k)M_{\alpha,m-1}^k = M_{\alpha,m}^k ,$$

which completes the mathematical induction for V_α. $M_{\alpha,n}^k \leq M_\alpha^k$ is clear since $M_{\alpha,n}^k$ is nondecreasing in n for each α and k. □

In the proof of Lemma 3.7, we also showed that if all the conditions in Lemma 3.7 hold, then for $i \in \mathcal{I}$ and $(s_0, \ldots, s_I) \in \mathcal{S}^{S-1}$ and for $(i, s_0, \ldots, s_I) \in \mathcal{S}_0^S$,

$$R_\alpha(i, s_0, \ldots, s_k, s_{k+1}+1, \ldots, s_I; n) - R_\alpha(i, s_0, \ldots, s_k+1, s_{k+1}, \ldots, s_I; n)$$

$$\leq (1-q_k)M^k_{\alpha,n} + (q_k - q_{k+1})U^{k+1}_\alpha$$

$$\leq (1-q_k)M^k_\alpha + (q_k - q_{k+1})U^{k+1}_\alpha, \qquad 0 \leq k \leq I,\ n \geq 1 . \qquad (3.4)$$

We can now state the main theorem.

Theorem 3.8. Assume the following conditions hold:

1. $C(i)$ is nondecreasing in i for $0 \leq i \leq I$.
2. $B(i)$ is nonnegative and nondecreasing in i for $0 \leq i \leq I$.
3. $A(i) - (C(i) + \frac{B(i)}{q_i} + \frac{1}{q_i}(P - \min\{A(0), C(0) + B(0)\}))$ is nondecreasing in i for $0 \leq i \leq I$.
4. $P \geq C(0) + B(0)$.
5. $0 < q_{k+1} \leq q_k \leq 1$ for $0 \leq k \leq I-1$.
6. $P_i(\cdot) \subset P_{i+1}(\cdot)$ for $0 \leq i \leq I-1$.

Then for each $0 \leq \alpha < 1$ there exists a stationary i control limit policy minimizing the total expected α-discounted cost for the model with spare units.

Proof: Note that Lemmas 3.5 through 3.7 hold since 3 of this theorem implies 5 of Lemma 3.5 with the assistance of conditions 1, 2, 4, and 5, and all the conditions of these lemmas are satisfied.

As has been seen twice in the previous chapter, if an optimality of an i control limit policy is guaranteed for the finite stage problem, the result can be easily extended for the infinite horizon case. Moreover, the stationarity of the i control limit policy optimizing the total α-discounted cost is guaranteed. For the n-stage problem, it is enough to show that for $(s_0,\ldots,s_I) \in \mathcal{S}^S$, $0 \le i \le I-1$ and $n \ge 1$,

$$f_n(i,s_0,\ldots,s_I) \le f_n(i+1,s_0,\ldots,s_I),$$

where

$$\begin{aligned} f_n(i,s_0,\ldots,s_I) = A(i) &+ \sum_{j=0}^{I} B(j)s_j + \alpha \sum_{i'=0}^{I} p_{ii'}R_\alpha(i',s_0,\ldots,s_I;n-1) \\ &- (C(i) + \sum_{j=0}^{I} B(j)s_j + B(i) \\ &+ \alpha R_\alpha(0,s_0,\ldots,s_i+1,\ldots,s_I;n-1)). \end{aligned}$$

Now, for $0 \le i \le I-1$, $(s_0,\ldots,s_I) \in \mathcal{S}^S$ and $n \ge 0$

$$\begin{aligned} &f_{n+1}(i+1,s_0,\ldots,s_I) - f_{n+1}(i,s_0,\ldots,s_I) \\ &\quad = A(i+1) - C(i+1) - B(i+1) - (A(i) - C(i) - B(i)) \\ &\qquad + \alpha(\sum_{i'=0}^{I} p_{i+1,i'}R_\alpha(i',s_0,\ldots,s_I;n) - \sum_{i'=0}^{I} p_{ii'}R_\alpha(i',s_0,\ldots,s_I;n)) \\ &\qquad - \alpha(R_\alpha(0,s_0,\ldots,s_{i+1}+1,\ldots,s_I;n) - R_\alpha(0,s_0,\ldots,s_i+1,\ldots,s_I;n)) . \end{aligned}$$

Here,

$$\sum_{i'=0}^{I} p_{i+1,i'} R_\alpha(i', s_0, \ldots, s_I; n) \geq \sum_{i'=0}^{I} p_{ii'} R_\alpha(i', s_0, \ldots, s_I; n) ,$$

because of Lemma 3.5, Lemma 1.5 and condition 6. By using (3.4), the last term can be simplified to

$$\alpha(R_\alpha(0, s_0, \ldots, s_{i+1}+1, \ldots, s_I; n) - R_\alpha(0, s_0, \ldots, s_i+1, \ldots, s_I; n))$$

$$\leq \alpha(1-q_i)M_\alpha^i + \alpha(q_i - q_{i+1})U_\alpha^{i+1}$$

$$\leq (1-q_i)M_1^i + (q_i - q_{i+1})U_1^{i+1} \text{ by 5 and since } M_\alpha^i \leq M_1^i,\ U_\alpha^{i+1} \leq U_1^{i+1}$$

$$= \frac{1-q_i}{q_i}(B(i+1) - B(i) + (q_i - q_{i+1})U_1^{i+1}) + (q_i - q_{i+1})U_1^{i+1}$$

$$= \frac{1-q_i}{q_i}(B(i+1) - B(i)) + \frac{q_i - q_{i+1}}{q_i q_{i+1}}(P + B(i+1) - \min\{A(0), C(0) + B(0)\})$$

$$= \left(\frac{1}{q_{i+1}} - 1\right) B(i+1) - \left(\frac{1}{q_i} - 1\right) B(i)$$

$$+ \left(\frac{1}{q_{i+1}} - \frac{1}{q_i}\right)(P - \min\{A(0), C(0) + B(0)\}) .$$

Therefore,

$$f_{n+1}(i+1, s_0, \ldots, s_I) - f_{n+1}(i, s_0, \ldots, s_I)$$

$$\geq A(i+1) - C(i+1) - B(i+1) - (A(i) - C(i) - B(i))$$

$$- \left[\left(\frac{1}{q_{i+1}} - 1\right) B(i+1) - \left(\frac{1}{q_i} - 1\right) B(i)\right.$$

$$\left. + \left(\frac{1}{q_{i+1}} - \frac{1}{q_i}\right)(p - \min\{A(0), C(0) + B(0)\})\right]$$

$$= A(i+1) - \left(C(i+1) + \frac{1}{q_{i+1}} B(i+1) + \frac{1}{q_{i+1}} (P - \min\{A(0), C(0)+B(0)\})\right)$$

$$- \left[A(i) - \left(C(i) + \frac{1}{q_i} B(i) + \frac{1}{q_i} (P - \min\{A(0), C(0)+B(0)\})\right)\right]$$

≥ 0, by 3.

This last inequality implies that for each $n \geq 1$ and $s = (s_0, \ldots, s_I) \in \mathcal{S}^S$, there exists an $i_{s,n}$ such that

$$\begin{cases} f_n(i, s_0, \ldots, s_I) < 0 & \text{for any } (i, s_0, \ldots, s_I) \text{ with } i < i_{s,n} \\ f_n(i, s_0, \ldots, s_I) \geq 0 & \text{for any } (i, s_0, \ldots, s_I) \text{ with } i \geq i_{s,n} \end{cases} .$$

Therefore, at the beginning of the n-stage α-discounted problem ($n \geq 1$), the decision to repair an operating machine if and only if its condition i is greater than or equal to some critical value $i_{s,n}$, where $(i,s) = (i, s_0, \ldots, s_I)$ is the state of the system, is optimal. This is an i control limit policy for the finite stage problem. A similar argument to that presented in Theorem 2.2 gives us the rest of what we need. □

In the previous chapter, we treated the model where there are no spare units and no explicit penalty cost. In order to show the existence of a control limit policy minimizing the total α-discounted cost, a condition was needed stating that the operating cost must increase more than the increase of the combined material and expected labor cost

for repairing a machine. Here, because of the presence of spare units and the explicit penalty cost, a stronger condition is required as is seen in condition 3. Since $P - \min\{A(0), C(0) + B(0)\}$ can be considered as a relative value of the penalty cost per period, 3 can be interpreted as follows: the operating cost must increase more than the increase of the combined material and expected labor costs and the maximal expected penalty cost effect for a single repair of a machine. Notice that the penalty effect vanishes if the repair time distribution does not depend on the condition of the machine to be repaired, i.e., if $q_k = q$ for any k. Other conditions are not very restrictive. From 1 and 2, it is evident that the worse the condition that an operating machine is in, the more expensive both the material and labor costs are for repairing it. From 5, the worse the condition that an operating machine is in, the longer the expected repair time is. 4 requires that the penalty cost is at least as expensive as the cheapest total repair cost of a single machine. The IFR property is guaranteed by 6.

The long-run average cost version of Theorem 3.8 can be easily obtained as before:

Theorem 3.9. Assume that all the conditions in Theorem 3.8 hold, and furthermore, suppose that any operating machine eventually fails. Then there exists a stationary i control limit policy minimizing the long-run expected average cost of the model with spare units.

Proof: If any operating machine eventually fails, then Theorem 1.4 holds since $(i, s_0, \ldots, s_I) = (0, 0, \ldots, 0)$ is accessible from every other state no matter which stationary policy is employed. The rest of the proof is similar to that of Theorem 2.3, and hence, can be omitted. □

3.2. Penalty Cost and Leasing Options

A penalty cost has been introduced in this chapter to distinguish the system's failure from the state where the system is operating. Notice that a system may also fail for the models without spare units. The reason why this penalty cost was not considered in Chapter 2 is that in those models the system's failure coincides with the machine's failure, and hence, the penalty cost for the system's failure can be included in the labor cost for repairing the machine. For the model with spare units, the failure of an operating machine does not necessarily mean the system's failure, and so this undesirable condition must be treated separately from other states, which requires the introduction of such a penalty cost.

If it is very undesirable to discontinue an operation, or a penalty cost for the system's failure is extremely expensive, and such a situation should be avoided, then a possible alternative would be to obtain a machine on lease from a leasing company when the necessity comes up. Suppose getting a machine on lease is allowed only when the system fails, and the cost for the lease is D per period. A leased machine continues to operate at its best condition during a leasing

period since the leasing company has the responsibility to replace older machines with new ones at its own expense whenever an operating leased machines become impercet. Therefore, the cost D includes the cost to maintain an operating machine in its best condition, and hence, it can be expensive. Thus, a decision to obtain a machine on lease would be practical only when the system fails.

Now consider the previous model with an option of leasing as stated above. The dynamic programming formulation is almost identical with an exception that every P which appeared in the formulation is replaced by $\min\{P,D\}$. Hence, it is clear that if $P > D$ then obtaining a machine on lease is optimal whenever the system fails and if $P \leq D$ then leasing had better be forgotten.

In the above discussion the leasing option is accepted only when the system fails. Now we can show that this statement is not an assumption to be made, but is a rule which an optimal policy must follow. To see this, let us allow the possibility that the leasing option is eligible at any time period. If the leasing option is chosen when there is an operating machine, it is practical to think that the operating machine replaced by a leased machine is brought to a repair shop no matter how good it is. This gives a dynamic programming formulation as below: For $n \geq 1$,

$$V_\alpha(i,s_0,\dots,s_I;n)$$

$$= \min\{A(i) + \sum_{j=0}^{I} B(j)s_j + \alpha \sum_{i'=0}^{I} \sum_{s_0'=0}^{s_0} \dots \sum_{s_I'=0}^{s_I} p_{ii'}$$

$$\cdot q^{(0)}_{s_0 s_0'} \dots q^{(I)}_{s_I s_I'} V_\alpha(i',s_0',\dots,s_I';n-1),\ C(i) + \sum_{j=0}^{I} B(j)s_j + B(i)$$

$$+ \alpha \sum_{s_0'=0}^{s_0} \dots \sum_{s_i'=0}^{s_i+1} \dots \sum_{s_I'=0}^{s_I} q^{(0)}_{s_0 s_0'} \dots q^{(i)}_{s_i+1, s_i'} \dots q^{(I)}_{s_I s_I'} V_\alpha(0,s_0',\dots,s_I';n-1),$$

$$D + C(i) + \sum_{j=0}^{I} B(j)s_j + B(i) + \alpha \sum_{s_0'=0}^{s_0} \dots \sum_{s_i'=0}^{s_i+1} \dots \sum_{s_I'=0}^{s_I} q^{(0)}_{s_0 s_0'} \dots q^{(i)}_{s_i+1, s_i'}$$

$$\dots q^{(I)}_{s_I s_I'} V_\alpha(0,s_0',\dots,s_I';n-1)\}$$

for $i \in \mathcal{I}$, $(s_0,\dots,s_I) \in \mathcal{S}^S$ and

$$V_\alpha(0,s_0,\dots,s_I;n)$$

$$= \min\{P,D\} + \sum_{j=0}^{I} B(j)s_j + \alpha \sum_{s_0'=0}^{s_0} \dots \sum_{s_I'=0}^{s_I}$$

$$\cdot q^{(0)}_{s_0 s_0'} \dots q^{(I)}_{s_I s_I'} V_\alpha(0,s_0',\dots,s_I';n-1)$$

for $(i,s_0,\dots,s_I) \in \mathcal{S}_0^{S+1}$.

Obviously for $i \in \mathcal{I}$, $(s_0,\dots,s_I) \in \mathcal{S}^S$, the third term which gives the minimum n period cost when the leasing option is chosen at the beginning is no less than the second term if $D \geq 0$, and hence, an optimal policy has the property that a leasing decision is not selected whenever there is an operating machine.

CHAPTER 4

COMPUTATIONAL RESULTS

This chapter treats the problem of finding the optimal cost and its corresponding policy knowing that an optimal policy is of a control limit form. Some recursive algorithms are presented for the total α-discounted cost case. For the long-run average cost case, an explicit calculation of an optimal control limit policy is obtained for systems with simple structures. In the last section some concluding remarks are given.

4.1. Algorithms for Calculating the Optimal Cost

As has been seen before, each model treated in this paper can be formulated as a Markov decision process. Therefore, the usual recursive algorithms to calculate the optimal policy for the total α-discounted cost or the long-run average cost can be applied to our models. These algorithms are mainly policy improvement procedures and linear programming approaches. Since these general techniques are studied in many papers, for example in [4], [8], [16], [19], we will not discuss them in detail here. In our models an optimal policy is assured to be of a control limit form if certain conditions are

satisfied. Knowing that an optimal policy is of a control limit form, we can expect that better algorithms exist since this information should enable us to explore this structure, thereby decreasing significantly the number of policies that must be considered.

The model treated here is the general Markov decision model defined in Sec on 1.3 with the action space $A = \{0,1\}$. 0 represents the action to keep a machine in operation, and 1 represents the action to repair it. Consider the total α-discounted cost problem. Suppose an optimal policy is known to be of a control limit form. Then by letting R_j be a control limit policy with a control limit j, the R_j's $(0 \leq j \leq I)$ are the candidates for the optimal policy. A simple enumeration is one possible method to find an optimal control limit policy though it does not seem to be very attractive. Total scanning is not necessary if the following lemma is used since it tells us that once $V_{R_k,\alpha}$ is obtained, the neighboring two values $V_{R_{k+1},\alpha}$ and $V_{R_{k-1},\alpha}$ can be easily checked to determine whether or not they are better than $V_{R_k,\alpha}$. Without having any ambiguity, we write V_{R_k} for $V_{R_k,\alpha}$ for the rest of this section and the next section.

Lemma 4.1. The following three expressions are equivalent.

For $i = 0, 1, \ldots, I-1$,

(a) $V_{R_i} \gtreqless V_{R_{i+1}}$.

(b) $C(i,0) + \alpha \sum_{j=0}^{I} p_{ij}(0) V_{R_i}(j) \lesseqgtr C(i,1) + \alpha \sum_{j=0}^{I} p_{ij}(1) V_{R_i}(j)$.

(c) $C(i,0) + \alpha \sum_{j=0}^{I} p_{ij}(0) V_{R_{i+1}}(j) \lesseqgtr C(i,1) + \alpha \sum_{j=0}^{I} p_{ij}(1) V_{R_{i+1}}(j)$.

Proof: We show that (a) and (b) are equivalent. The equivalence between (a) and (c) can be similarly shown.

A control limit policy $R_i = (f_i, f_i, \ldots)$ has the form

$$f_i(j) = \begin{cases} 0 & \text{for } 0 \leq j < i \\ 1 & \text{for } i \leq j \leq I, \end{cases}$$

and from Markov decision theory, V_{R_i} satisfies the following equation.

$$V_{R_i}(j) = C(j, f_i(j)) + \alpha \sum_{j'=0}^{I} p_{jj'}(f_i(j)) V_{R_i}(j'),\ 0 \leq j \leq I. \tag{4.1}$$

Now let $C_{R_i}(j) = C(j, f_i(j))$, $0 \leq j \leq I$, and let P_{R_i} be the transition probability matrix modified by the policy R_i. Then we can solve (4.1) for V_{R_i} and obtain

$$V_{R_i} = (I - \alpha P_{R_i})^{-1} C_{R_i} .$$

Using this expression and from simple calculations, we have

$$V_{R_i} - V_{R_{i+1}} = [C(i,1) - C(i,0) + \alpha(\sum_{j=0}^{I} (p_{ij}(1) - p_{ij}(0)) V_{R_i}(j))]$$
$$\cdot [(i+1)\text{-st column of } (I - \alpha P_{R_{i+1}})^{-1}] .$$

As each column of $(I - \alpha P_{R_{i+1}})^{-1}$ is nonnegative and has at least one positive element, the claim follows. □

Definition. V_{R_i} is said to be unimodal in i $(0 \leq i \leq I)$ if $V_{R_i} < V_{R_{i-1}}$ implies $V_{R_{i-1}} < V_{R_{i-2}}$ $(2 \leq i \leq I)$ and $V_{R_i} < V_{R_{i+1}}$ implies $V_{R_{i+1}} < V_{R_{i+2}}$ $(0 \leq i \leq I-2)$.

If V_{R_i} is unimodal in i, then Fibonacci type search can be applied resulting in a faster convergence rate.

Policy improvement procedures can also be improved if V_{R_i} is unimodal in i. If V_{R_i} is unimodal in i, then the technique of the policy improvement algorithm can be performed among the set of control limit policies rather than the set of whole stationary policies. Therefore the convergence rate of the special policy improvement technique shown next should be much faster than the general policy improvement technique. The algorithm is as follows:

<u>Step 0</u>. Find a control limit policy $R_{i_0} = (f_{i_0}, f_{i_0}, \ldots)$.

<u>Step 1</u>. Compute the value $V_{R_{i_0}}$ by solving (4.1) or for $0 \le j \le I$,

$$V_{R_{i_0}}(j) = C(j, f_{i_0}(j)) + \alpha \sum_{j'=0}^{I} p_{jj'}(f_{i_0}(j)) V_{R_{i_0}}(j') .$$

<u>Step 2</u>. Let

$$S_1 = \{0 \le i' < i_0 \mid C(i,1) + \alpha \sum_{j=0}^{I} p_{ij}(1) V_{R_{i_0}}(j) < V_{R_{i_0}}(i)$$

$$\text{for any } i \text{ with } i' \le i \le i_0 - 1\}$$

$$S_2 = \{i_0 < i' \le I \mid C(i,0) + \alpha \sum_{j=0}^{I} p_{ij}(0) V_{R_{i_0}}(j) < V_{R_{i_0}}(i)$$

$$\text{for any } i \text{ with } i_0 \le i \le i' - 1\} .$$

Take i_1 to be the smallest $i' \in S_1$ or the largest $i' \in S_2$. If $S_1 = S_2 = \emptyset$, then R_{i_0} is optimal. Otherwise, a control limit policy $R_{i_1} = (f_{i_1}, f_{i_1}, \ldots)$ is strictly better than R_{i_0}. Go back to Step 1 where R_{i_1} replaces R_{i_0}.

Notice that if R_{i_0} is not optimal, then either $V_{R_{i_0-1}} < V_{R_{i_0}}$ or $V_{R_{i_0+1}} < V_{R_{i_0}}$. By Lemma 4.1, the former leads to $i_0-1 \in S_1$ and the latter leads to $i_0+1 \in S_2$. Hence, if R_{i_0} is not optimal, then $S_1 \cup S_2 \neq \emptyset$. Also notice that if $i_1 \in S_1 \cup S_2$, then by the definition of S_1 and S_2,

$$T_{R_{i_1}} V_{R_{i_0}} = C_{R_{i_1}} + \alpha P_{R_{i_1}} V_{R_{i_0}} \lneqq V_{R_{i_0}} .$$

Using the monotonicity of $T_{R_{i_1}}$ (see Ross [19]), we have

$$V_{R_{i_1}} = \lim_{n \to \infty} T^n_{R_{i_1}} V_{R_{i_0}} \lneqq V_{R_{i_0}} .$$

Therefore, R_{i_1} is a strict improvement.

The problem thus becomes that of determining under what conditions the unimodality of V_{R_i} is assured. The following theorem holds:

<u>Theorem 4.2</u>. V_{R_i} is unimodal in i $(0 \leq i \leq I)$ under the following conditions.

1. $C(i,1)$ is nondecreasing in i for $0 \leq i \leq I$.
2. $C(i,0) - C(i,1)$ is nondecreasing in i for $0 \leq i \leq I$.
3. $P_i(\cdot) \subset P_{i+1}(\cdot)$ for $0 \leq i \leq I-1$

   where

$$P_i(j) = \sum_{k \leq j} p_{ik}(0).$$

4. $p_{ij}(1) = p_{i+1,j}(1)$ for $0 \leq i \leq I-1$, $0 \leq j \leq I$.
5. $p_{ij}(0) = 0$ for $0 \leq j \leq i-1$.

Proof: First notice that the first three conditions guarantee the existence of a control limit policy optimizing the model.

We want to show that $V_{R_{i+1}} > V_{R_i}$ implies $V_{R_{i+2}} > V_{R_{i+1}}$ for $0 \leq i \leq I-2$. Suppose $V_{R_{i+1}} > V_{R_i}$. Then for $j \geq i+1$,

$$V_{R_{i+1}}(j) = C(j,1) + \alpha \sum_{k=0}^{I} p_{jk}(1)V_{R_{i+1}}(k), \text{ by (4.1)}$$

$$\leq C(j+1,1) + \alpha \sum_{k=0}^{I} p_{j+1,k}(1)V_{R_{i+1}}(k) \text{ , from 4 and 1}$$

$$= V_{R_{i+1}}(j+1), \text{ by (4.1).}$$

Hence, $V_{R_{i+1}}(j)$ is nondecreasing in j for $j \geq i+1$. Furthermore,

$$V_{R_{i+1}}(i) - V_{R_{i+1}}(i+1)$$

$$= C(i,0) + \alpha \sum_{j=0}^{I} p_{ij}(0)V_{R_{i+1}}(j) - (C(i+1,1) + \alpha \sum_{j=0}^{I} p_{i+1,j}(1)V_{R_{i+1}}(j))$$

$$\leq C(i,0) + \alpha \sum_{j=0}^{I} p_{ij}(0)V_{R_{i+1}}(j) - (C(i,1) + \alpha \sum_{j=0}^{I} p_{ij}(1)V_{R_{i+1}}(j)),$$

by 1 and 4

$$\equiv \theta \geq 0, \text{ from } V_{R_{i+1}} > V_{R_i} \text{ and by Lemma 4.1(a).}$$

Then

$$C(i+1,0) + \alpha \sum_{j=0}^{I} p_{i+1,j}(0)V_{R_{i+1}}(j) - (C(i,0) + \alpha \sum_{j=0}^{I} p_{ij}(0)V_{R_{i+1}}(j))$$

$$= C(i+1,0) - C(i,0) + \alpha \sum_{j=i+1}^{I} p_{i+1,j}(0)V_{R_{i+1}}(j) - \alpha(p_{ii}(0)V_{R_{i+1}}(i+1)$$

$$+ p_{i,i+1}(0)V_{R_{i+1}}(i+1) + p_{i,i+2}(0)V_{R_{i+1}}(i+2) + \cdots + p_{iI}(0)V_{R_{i+1}}(I))$$

$$- \alpha(p_{ii}(0)V_{R_{i+1}}(i) - p_{ii}(0)V_{R_{i+1}}(i+1))$$

$$\geq C(i+1,0) - C(i,0) - \alpha p_{ii}(0)(V_{R_{i+1}}(i) - V_{R_{i+1}}(i+1)),$$

since $V_{R_{i+1}}(i+1)$, $V_{R_{i+1}}(i+1)$, $V_{R_{i+1}}(i+2), \ldots, V_{R_{i+1}}(I)$ is nondecreasing, and by 3

$$\geq C(i+1,0) - C(i,0) - \alpha\theta.$$

Finally,

$$C(i+1,0) + \alpha \sum_{j=0}^{I} p_{i+1,j}(0) V_{R_{i+1}}(j) - (C(i+1,1) + \alpha \sum_{j=0}^{I} p_{i+1,j}(1) V_{R_{i+1}}(j))$$

$$= (C(i+1,0) + \alpha \sum_{j=0}^{I} p_{i+1,j}(0) V_{R_{i+1}}(j)) - (C(i,0) + \alpha \sum_{j=0}^{I} p_{ij}(0) V_{R_{i+1}}(j))$$

$$+ (C(i,0) + \alpha \sum_{j=0}^{I} p_{ij}(0) V_{R_{i+1}}(j)) - (C(i,1) + \alpha \sum_{j=0}^{I} p_{ij}(1) V_{R_{i+1}}(j))$$

$$+ C(i,1) - C(i+1,1), \text{ by } 4$$

$$\geq C(i+1,0) - C(i,0) - \alpha\theta + \theta + C(i,1) - C(i+1,1) > 0, \text{ from } 2.$$

Therefore, by Lemma 4.1(a), we have $V_{R_{i+1}} > V_{R_{i+1}}$. In a similar manner we can prove that $V_{R_i} < V_{R_{i-1}}$ implies $V_{R_{i-1}} < V_{R_{i-2}}$, which completes the proof of unimodality of V_{R_i} in i $(0 \leq i \leq I)$. □

Condition 5 says that a machine only deteriorates and never gets better while it keeps operating, which seems to be a reasonable assumption. Condition 4 is satisfied for the classical replacement models since $p_{i0}(1) = 1$ for any i. Also, in our model treated in Section 2.1, condition 4 is automatically satisfied. Further, if, in addition to the conditions in Theorem 2.2, we assume $p_{ij} = 0$ for any $j \leq i-1$, the unimodality of V_{R_i} in i is guaranteed. For other models, condition 4 seems too restrictive since it requires the repair time distribution of a machine to be independent of its operating condition.

4.2. Policy Improvement Hybrid Method

When a policy improvement technique is employed, the number of feasible policies can be significantly reduced if better policies are searched recursively among a set of stationary control limit policies. As has been just seen, this can be achieved if unimodality of V_{R_i} in i is assured, and unimodality is satisfied under mild conditions for replacement or simple structured repair models. For more complicated models, unimodality can no longer be expected. Then an optimal policy may or may not be obtained iteratively by searching better policies among a set of stationary control limit policies. But it is conceivable to think that we can still proceed efficiently if we construct an algorithm where "good" policies are searched iteratively among stationary control limit policies whenever possible (first phase), and then we switch to a usual policy improvement among stationary policies (second phase). The second phase starts with a policy which is better than the control limit policy obtained at the end of the first phase if the first phase does not yield an optimal policy. Thus a hybrid algorithm is introduced and is indicated in the flow chart of Fig. 4.1.

Step 0. Find a control limit policy $R^0_{i_0} = (f_{i_0}, f_{i_0}, \ldots)$.

Step 1. Compute the value $V_{R^0_{i_0}} \equiv V_{f_{i_0}}$ by solving

$$V_{f_{i_0}}(j) = C(j, f_{i_0}(j)) + \alpha \sum_{j'=0}^{I} p_{jj'}(f_{i_0}(j)) V_{f_{i_0}}(j'), \quad 0 \le j \le I.$$

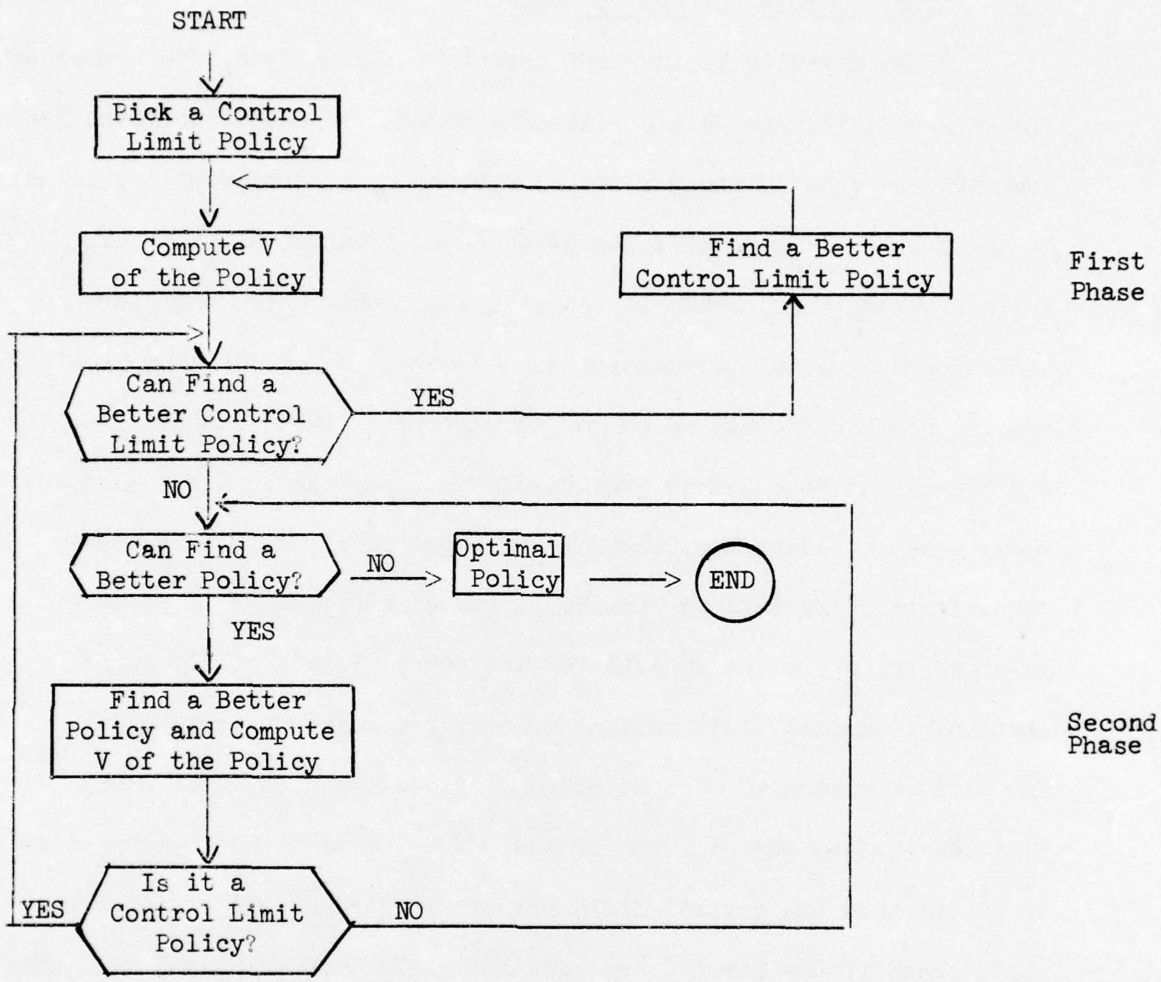

Figure 4.1: Flow Chart of Policy Improvement Hybrid Algorithm.

Step 2. Let

$$S_1 = \{0 \leq i' < i_0 | C(i,1) + \alpha \sum_{j=0}^{I} p_{ij}(1) V_{f_{i_0}}(j) < V_{f_{i_0}}(i)$$

$$\text{for any } i \text{ with } i' \leq i \leq i_0 - 1\}$$

$$S_2 = \{i_0 < i' \leq I | C(i,0) + \alpha \sum_{j=0}^{I} p_{ij}(0) V_{f_{i_0}}(j) < V_{f_{i_0}}(i)$$

$$\text{for any } i \text{ with } i_0 \leq i \leq i'-1\} .$$

Take i_1 to be the smallest $i' \in S_1$ or the largest $i' \in S_2$. If $S_1 \cup S_2 \neq \emptyset$, i_1 exists and $R^1_{i_1} = (f_{i_1}, f_{i_1}, \ldots)$ is strictly better than $R^0_{i_0}$, and go back to Step 1 where $R^1_{i_1}$ is now a new $R^0_{i_0}$. If $S_1 = S_2 = \emptyset$, go to Step 3.

Step 3. Let $R^2 = (f,f\ldots)$ be such that

$$C(i,f(i)) + \alpha \sum_{j=0}^{I} p_{ij}(f(i)) V_{f_0}(j)$$

$$= \min_{a=0,1} \{C(i,a) + \alpha \sum_{j=0}^{I} p_{ij}(a) V_{f_{i_0}}(j)\}, \quad 0 \leq i \leq I.$$

Compute the value $V_{R^2} \equiv V_f$ by solving

$$V_f(i) = C(i,f(i)) + \alpha \sum_{j=0}^{I} p_{ij}(f(i)) V_f(j), \quad 0 \leq i \leq I .$$

If V_f is not an improvement, R^2 is optimal. If V_f is an improvement, then go to Step 4.

Step 4. If $R^2 = R^2_{i_2}$, or if R^2 is a control limit policy, then go to Step 2 where $R^2_{i_2}$ is now a new $R^0_{i_0}$, otherwise go to Step 3 where the present V_f is now a new V_{f_0} and R^2 is renewed.

The justification of the above algorithm is similar to that of the previous algorithm in the last section. The algorithm is presented for the models without spare units, but the hybrid algorithm can be applied to the models with spare units with a slight notational change.

4.3. Explicit Calculation of the Optimal Control Limits

Knowing that an optimal policy is of a control limit form, we found some recursive algorithms which produce an optimal control limit for the total α-discounted cost problem. When we move on to the long-run average cost problem, we find that both policy improvement techniques and LP approaches also are applicable to that case. Moreover, if the model has a simple structure described as below, we can even calculate the optimal control limit explicitly. Explicit calculation makes it easier to grasp the effect of several cost coefficients on the determination of a control limit. In this section, we consider a few examples and find an explicit way of calculating an optimal control limit in each model. We require simplifying assumptions on the transition probabilities, the linearity of several costs, and the linearity of the expected repair time length.

Example 4.1. We consider a maintenance model without spare units. The transition probability p_{ij}, the parameter q_i of repair time distribution, and the cost functions $A(i)$, $C(i)$ and $B(i)$ are defined as follows:

$$p_{ij} = \begin{cases} 1-p & \text{if } j = i \\ p & \text{if } j = i+1, \qquad 0 \le i \le I-1. \\ 0 & \text{otherwise} \end{cases}$$

$$q_i = q > 0 , \qquad 0 \le i \le I.$$

$$A(i) = a_0 + ai , \qquad 0 \le i \le I.$$

$$C(i) = c_0 + ci , \qquad 0 \le i \le I .$$

$$B(i) = b_0 + bi , \qquad 0 < i < I.$$

The long-run average cost using a control limit policy R_i (i is its control limit) can be calculated as the ratio of the expected cost between returns to state 0 when R_i is employed to the expected length of a return to state 0 given R_i is employed. In this example, by letting $1/p \equiv \lambda_0$, $1/q \equiv \mu_0$ for simplicity,

$$V_{R_i} = \frac{\lambda_0(a_0 i + \frac{a}{2} i(i-1)) + c_0 + ci + \mu_0(b_0 + bi)}{\lambda_0 i + \mu_0} .$$

Taking the derivative of V_{R_i} with respect to i, setting the value of the derivative to be zero, and solving it for i (≥ 0) yields,

$$i_{opt} = -\frac{\mu_0}{\lambda_0} + \frac{1}{\lambda_0}\sqrt{\mu_0(\mu_0+\lambda_0) + \frac{2}{a}(\mu_0\lambda_0(b_0 - a_0) + \lambda_0 c_0 - \mu_0 c - \mu_0^2 b)} .$$

Let i^* be the optimal control limit. Then it is easy to see that if the term inside of the square root of the above expression is negative or if $i_{opt} \leq 0$, then $i^* = 0$. A simple calculation shows that

$$a_0 - \frac{a}{2} - b_0 + \frac{p}{q} b - qc_0 + pc > 0$$

guarantees $i^* = 0$. For other cases, if $i_{opt} < I$, then $i^* = [i_{opt}]$ or $[i_{opt} + 1]$, and if $i_{opt} \geq I$, then $i^* = I$.

Example 4.2. We generalize Example 4.1 by letting

$$\frac{1}{q_i} \equiv \mu_i = \mu_0 + \mu i, \quad \mu_0 \geq 1, \quad \mu \geq 0 \quad \text{for} \quad 0 \leq i \leq I.$$

Then,

$$V_{R_i} = \frac{\lambda_0\left(a_0 i + \frac{a}{2} i(i-1)\right) + c_0 + ci + (\mu_0 + \mu i)(b_0 + bi)}{\lambda_0 i + (\mu_0 + \mu i)} .$$

The same analysis as in Example 4.1 gives,

$$i_{opt} = -\frac{\mu_0}{\lambda_0 + \mu} + \frac{\sqrt{D}}{(\lambda_0 + \mu)\sqrt{\lambda_0 a + 2\mu b}} ,$$

where

$$D = (\lambda_0 + \mu)[\lambda_0\mu_0(a + 2(b_0 - a_0)) + 2(\lambda_0 + \mu)c_0 - 2\mu_0 c] + \lambda_0\mu_0^2(a - b).$$

Example 4.3. If we further generalize Example 4.2 by letting

$$p_{ij} = \begin{cases} 1-p_i & \text{if } j = i \\ p_i & \text{if } j = i+1 \\ 0 & \text{otherwise} \end{cases}$$

for $0 \leq i \leq I-1$, where

$$\frac{1}{p_i} \equiv \lambda_i = \lambda_0 + \lambda i \ , \qquad 0 \leq i \leq I-1,$$

then the expression gets much more complicated, and $i = i_{opt}$ is the solution of the following polynomial of degree 4:

$$\frac{\lambda^2 a}{6}\, i^4 + \frac{2\lambda\gamma a}{3}\, i^3 + \left(\frac{\alpha\gamma}{2} - \frac{\lambda\beta}{2} + \lambda\mu_0 a\right) i^2$$

$$+ (\alpha\mu_0 - \lambda c_0 - \lambda\mu_0 b_0) i + (\beta\mu_0 - c_0\gamma - \mu_0\gamma b_0) = 0 \ ,$$

where

$$\alpha = \lambda a_0 + \lambda_0 a - \lambda a + 2\mu b$$

$$\beta = \lambda_0 a_0 - \frac{1}{2}\lambda a_0 - \frac{1}{2}\lambda_0 a + \frac{1}{6}\lambda a + c + \mu b_0 + \mu_0 b$$

$$\gamma = \lambda_0 - \frac{1}{2}\lambda + \mu.$$

In this example, the optimality of a control limit policy is guaranteed if c, b and μ are nonnegative, $c_0 = 0$ and

$$a \geq c + \mu b_0 + \mu_0 b + (2I - 1)\mu b.$$

Example 4.4. We generalize Example 4.1 by allowing an operating machine to transit into a failed state in a period. In this example, an operating machine will fail to operate in one period with probability r (> 0), and if not, then it goes to the next worse state with probability p, or it remains in the same state with probability $1-p$.

An optimality of a control limit policy is guaranteed if $a \geq c \geq 0$. Assuming this condition, it is not hard to see that V_{R_i} is given by the following expression:

$$V_{R_i} = \frac{\frac{1}{r}\,[a_0(1-d)^i + a(\frac{d-d^i}{1-d} - (i-1)d^i)] + (c(i-I) + \frac{b}{q}\,(i-I))d^i}{\frac{1}{r}\,(1-d^i) + \frac{1}{q}},$$

where

$$0 \leq d = \frac{p - pr}{p + r - pr} < 1 .$$

Taking the derivative of V_{R_i} with respect to i and setting it to be zero yields,

$$\Omega(i) \equiv \frac{1}{r}\,(\frac{a}{r} - \gamma)d^i - (\log d)(\frac{1}{r} + \frac{1}{q})(\frac{a}{r} - \gamma)i - (\frac{a}{r} - \gamma)(\frac{1}{r} + \frac{1}{q})$$
$$- (\log d)\,(\frac{a_0}{qr} - \frac{\gamma_0}{r} + \frac{p(1-r)a}{qr^2} + \frac{\gamma I}{q}) = 0,$$

where

$$\gamma = c + \frac{b}{q}, \quad \gamma_0 = c_0 + \frac{b_0}{q} .$$

If $a/r > \gamma$, we can show that $d\Omega(i)/di > 0$. Then $\Omega(i)$ has at most one zero point, and if we let $\Omega(i_{opt}) = 0$ when there is a zero point of $\Omega(i)$, $i^* = [i_{opt}]$ or $[i_{opt} + 1]$ for $0 < i_{opt} < I$, $i^* = 0$ for $i_{opt} \leq 0$, and $i^* = I$ for $i_{opt} \geq I$. If there is no zero point, then $i^* = 0$.

If the failure rate r is small enough, then d is close to 1, and hence, we can approximate the zero point of $\Omega(i)$ by using the first degree of approximation $d^i = e^{(\log d)i} = 1 + (\log d)i$. Then,

$$i_{opt} = - \frac{1}{\log d} - \frac{ra_0 - rq\gamma_0 + p(1-r)a + r^2\gamma I}{r(a - r\gamma)} .$$

4.4. Conclusion

As was noted in the previous chapters, optimality of a control limit policy is guaranteed under certain mild conditions in each maintenance model with repair. The main restrictions are placed on the cost functions. In the case of replacement models, the nondecreasing property of the difference of the operating cost and the material cost is required. If the repair models without spare units are of interest, the nondecreasing property of the difference of the operating cost and the total repair cost for a single repair work, including the material and labor costs is required. When the repair model with spare units is treated, this condition is strengthened in the sense that the total repair cost now includes the effect of the penalty cost as well as the material and labor costs.

For the model with spare units, we assume that there are a large enough number of repairmen in the repair shop so that the repair work can begin immediately on any machine sent to the repair shop. Having only one repairman in the repair shop also seems to be a reasonable assumption. Then at most one machine can be repaired at a time, and if more than one machine is to be repaired, they must form a queue while they wait for service. This modified problem results in a slightly different formulation, but it can be shown that almost identical conditions as presented in this paper will be sufficient for the optimality of an i control limit policy.

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20. Abstract

In this study discrete time finite state Markov maintenance models are investigated. In each model, a machine is assumed to be operating over time with its condition deteriorating as time goes on. The state of the machine is observed at the beginning of a period. An operating machine can be sent to a repair shop at this time, whereas a failed machine must be repaired. When a machine is being repaired, the number of time periods that it is unavailable is usually assumed to have a geometric distribution. A repaired machine becomes available in its best state. An operating cost is charged while a machine is operating, and material and labor costs are charged when it is being repaired. The objective is to find a policy which minimizes the total expected α-discounted cost or the long-rung average cost. Special emphasis is being placed on finding sufficient conditions to assure that a control limit policy is optimal. The aforementioned model had only one machine in the system. Models with spare machines in the system are next studied. For these models a penalty cost is added when the system fails (only when all machines are inoperative).

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